Economics of Information and Uncertainty

Summer Term of 2006

1 Introduction

STATIC DECISION POROBLEMS

1.1 Various Informational Settings of Decision Problems

1. Decision problems under certainty

Agent's choice variables (actions) lead to a *certain* payoff		
a_1	$\rightarrow$	u_1
a_2	$\rightarrow$	u_2
.	$\rightarrow$	.
.	$\rightarrow$	.

$\rightarrow$ cf. e.g. classic consumer or producer theory (micro lecture)

2. Strategic Interdependencies

Actions	b_1	b_2	..
a_1	(u_{11}, v_{11})	(u_{12}, v_{12})	..
a_2	(u_{21}, v_{21})	(u_{22}, v_{22})	..
.	.	.	.

$\rightarrow$ Game Theory analyzes situations where an agent's payoff may not solely depend on her own actions but also (in part) on another agent's actions (e.g. the utility one draws from dining with a friend may depend on the dress she's wearing)

3. Uncertainty

States of the world/actions	z_1	z_2	..
a_1	(u_{11})	(u_{12})	..
a_2	(u_{21})	(u_{22})	..
.	.	.	.

Decisions under uncertainty can be modeled as games where the player that draws first (nature) does not act strategically but picks her "action" according to a given probability distribution. That probability distribution is usually assumed to be common knowledge amongst all the other players (inter alia to make sure their beliefs are consistent). Nature's actions are commonly refered to as "states of the world".

Example: Choice of career

(a) Bavarian Gov't worker: 5.000,-

(b) Entrepreneur: 20.000,- : z_1
or: 1.000,- : z_2

Lottery: L1 = (1 , 0 ; 5.000 , 0)
L2 = (p , 1-p ; 20.000 , 1.000)

4. Strategic interdependencies under uncertainty

- Principal-Agent-issues

→ Economics of Information: The principal's payoff may not be solely contingent on the agent's effort choice but may also be subject to the whims of Fortune (e.g. firm owner's profit).

1.2 Main Elements of our Analysis

(a) Choice Variables (Actions) $A = \{a_1, ..., a_n\}$

(b) States of the World $Z = \{z_1, ..., z_m\}$

(c) Probability Vectors $P = \{p^1, ..., p^n\}$ with $p^j = (p_1^j, ..., p_m^j)$
exhibiting the following properties: $0 \leq p_i^j \leq 1 \forall i, j$ $\Sigma_{i=1}^m p_i^j = 1 \forall i$

(d) Payoff Vectors $X = \{x^1, x^2, ..., x^n\}$ with $x^j = (x_{j1}, x_{j2}, ..., x_{jm})$
Set of Lotteries: $L = \{L_1, ..., L_n\}$ $L_j := (p^j : x^j)$
What we're after: $a^\star \in A$, that will optimize ..?...
In order for us to be able to solve this decision problem, we need a preference ordering over lotteries! We'll now throw a glance at a few possible candidates.

1.3 Pros & Cons of Some Possible Decision Criteria

1. Expected Value of a Lottery

$$L_1 \succeq L_2 \Leftrightarrow \mu_1 = \Sigma p_i^1 x_{1i} \geq \mu_2 = \Sigma p_i^2 x_{2i}$$

pro: Under certain circumstances, Evolution may favor expected value maximizing individuals.
con:

$$L_1 = (1/2, 0, 1/2; 10.000, 5.000, 10)$$

$$L_2 = (0, 1, 0; 10.000, 5.000, 10)$$

$$\mu_1 = 5.005 > \mu_2 = 5.000$$

Risk is being ignored.
Utility function: $U(L_j) = \mu_j$
(e.g. risk-neutral firms, Gov't)

2. The Maximin-Criterion

$$L_1 \geq L_2 \Leftrightarrow min_i[x_{1i}|p_i^1 > 0] \geq min_i[x_{2i}|p_i^2 > 0]$$

pro: takes account of losses/ unfavorable outcomes
con: L_1 =(0,99 , 0,01 ; 10.000 , 0)
L_2=(0 , 1 ; 10.000 , 5.000)

$$\Rightarrow L_2 \geq L_1$$

Problem: Losses may be over-emphasized (unlikely though they may be)
Utility: $U(L_j) = min_i[x_{ji}|p_i^j > 0]$
(*very* risk-averse indivduals)

3. The $\mu - \sigma$-Criterion
Basic idea: Agents will like a big payout (μ), but they will dislike risk (σ)
Utility: $U(L_j) = \mu_j - k\sigma_j$ $[= f(\mu_j, k\sigma_j)]$
$\mu_j = \Sigma_i p_i^j x_{ji}$
$\sigma_j^2 = \Sigma_i p_i^j (x_{ji} - \mu_j)^2$
(k is a measure of agent's risk-aversion)

pro: - rather intuitive; easy to handle

con: - Further moments of the distribution, such as (e.g.) curtosis, are being ignored.

Example:
$L_1 = (50/100, 49/100, 1/100; 102, 100, 0)$
$L_2 = (1/2, 1/2; 100 + 102, 100 - 102)$
$\Rightarrow \mu_1 = \mu_2 = 100$, $\sigma_1 = \sigma_2 = 102$
$\Rightarrow L_1 \sim L_2$
!!! This criterion is commonly used in capital markets theory (CAPM)

4. Expected Utility
   Swiss mathematician Daniel Bernoulli (1738) came up with a solution to the *St.Petersburg Paradox*:

   - Toss a coin time and again for as long as it is showing tails;

   - As soon as it is showing heads, the game is over and the gambler, who bought into the bet, will get a pay-out of 2^{n+1}, where n is the number of times the coin had previously been showing tails.

   Expected value of pay-out:
   $\mu = 1/2 * 2 + 1/2(1/2 * 2^2) + 1/2 * 1/2 * 1/2 * 2^3 + ... = 1 + 1 + 1 + ... = +\infty$
   Bernoulli's proposal: u(x) = ln(x)
   $\rightarrow U(L) = 1/2 * ln(2) + 1/4 * ln(4) + ... < +\infty$

   $$U(L_j) = \Sigma_i p_i^j u(x_{ji})$$

   Looks swell alright, but can this criterion be generalized??

   This question we'll endeavor to answer in the following chapter.

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2 Expected Utility Theory

In their seminal work "Theory of Games and Economic Behavior" (1944), John von Neumann and Oscar Morgenstern develop the axiomatic foundations of Expected Utility Theory.[1]We shall first expose the axioms (2.1.), from which we will then derive the pivotal vNM-theorem (2.2.). We shall conclude by looking at some basic properties of vNM-utility functions (2.3.).

2.1 The Axioms

Let L be a set of lotteries

$\{L_1, ..., L_n\} \equiv L.$

Let there be a "standard lottery" $(1-u, u; x_{min}, x_{max})$
where $u = Prob(x_{max})$, and where x_{min} and x_{max} be chosen in such a way that the following weak inequalities hold:

$x_{min} \leq x \ \forall x \in X$ and $x_{max} \geq x \ \forall x \in X$

where X is the matrix consisting of the pay-off vectors X_i pertaining to lotteries $L_i \in L$

1. Axiom 1: Ordering of Lotteries

   Completeness:
   $\forall (L_i, L_j) \in (L \times L) : L_i \succeq L_j \vee L_j \succeq L_i$
   i.e. For any two given lotteries, an individual will always be able to tell which one she likes better.

   Transitivity:
   $\forall (L_i, L_j, L_k) \in (L \times L \times L)$

[1]Von Neumann, J. and O.Morgenstern; Theory of Games and Economic Behavior; Princeton, N.J.; Princeton University Press

$(L_i \succeq L_j \wedge L_j \succeq L_k) \Longrightarrow L_i \succeq L_k$
i.e. If agent likes oranges better than apples and apples better than pears, we can infer from that that she likes oranges better than pears, too.

Reflexivity:
$\forall L_i \in L : L_i \succeq L_i$
i.e. 1 lb of apples is no worse than 1 lb of apples.

This first axiom is sometimes refered to as the "rationality axiom". It is perfectly analogous to similar axioms in standard micro theory under certainty.

2. Axiom 2: Preferences over Probabilities

   Let there be standard lotteries $L_i \in L$ with $L_i = (1 - u_i, u_i; x_{min}, x_{max})$
   $L_1 \succeq L_2 \Longleftrightarrow u_1 \geq u_2$

   This axiom is very much akin to the axiom of local non-satiation, which we know from standard consumer theory. It says that, given a choice between two standard lotteries, agent will prefer the one with more probability mass on x_{max}.

3. Axiom 3: Continuity

   $\forall x \in [x_{min}; x_{max}] : \exists u(x) \in [0; 1]$ s.d.
   $x \sim (1 - u(x), u(x); x_{min}, x_{max})$

   This says that for any given lottery, it is always possible to construct a standard lottery such that agent be indifferent between the two.

   Example:
   $x_{min} = 0$, $x_{max} = 10.000$, $x = 1.000$
   In that case, agent is indifferent between getting a certain payment of 1.000 or getting 10.000 with probability u(1.000).

4. Axiom 4: Independence

   $\forall (L_i, L_j, L_k) \in (L \times L \times L)$ with $L_i \succeq L_j$ and $\forall \omega \in [0; 1]$:

   $(1 - \omega, \omega; L_i, L_k) \succeq (1 - \omega, \omega; L_j, L_k)$

This looks rather plausible: With both lotteries agent will get L_k with probability ω. But with the first lottery, she'll get L_i with probability $1 - \omega$, whereas with the same probability she'll only get L_j (which, by hypothesis, is worse than L_i) with the second lottery. Thus, we'd assume she'd rather have the first lottery. Empirical findings suggest, however, that this independence axiom may in some instances be problematic (cf. last chapter of this lecture). Indeed, the axiom presupposes that:

- Agents can handle compound lotteries (i.e. lotteries over lotteries)

- Agents are aware that there are no complement effects between lotteries

For example:

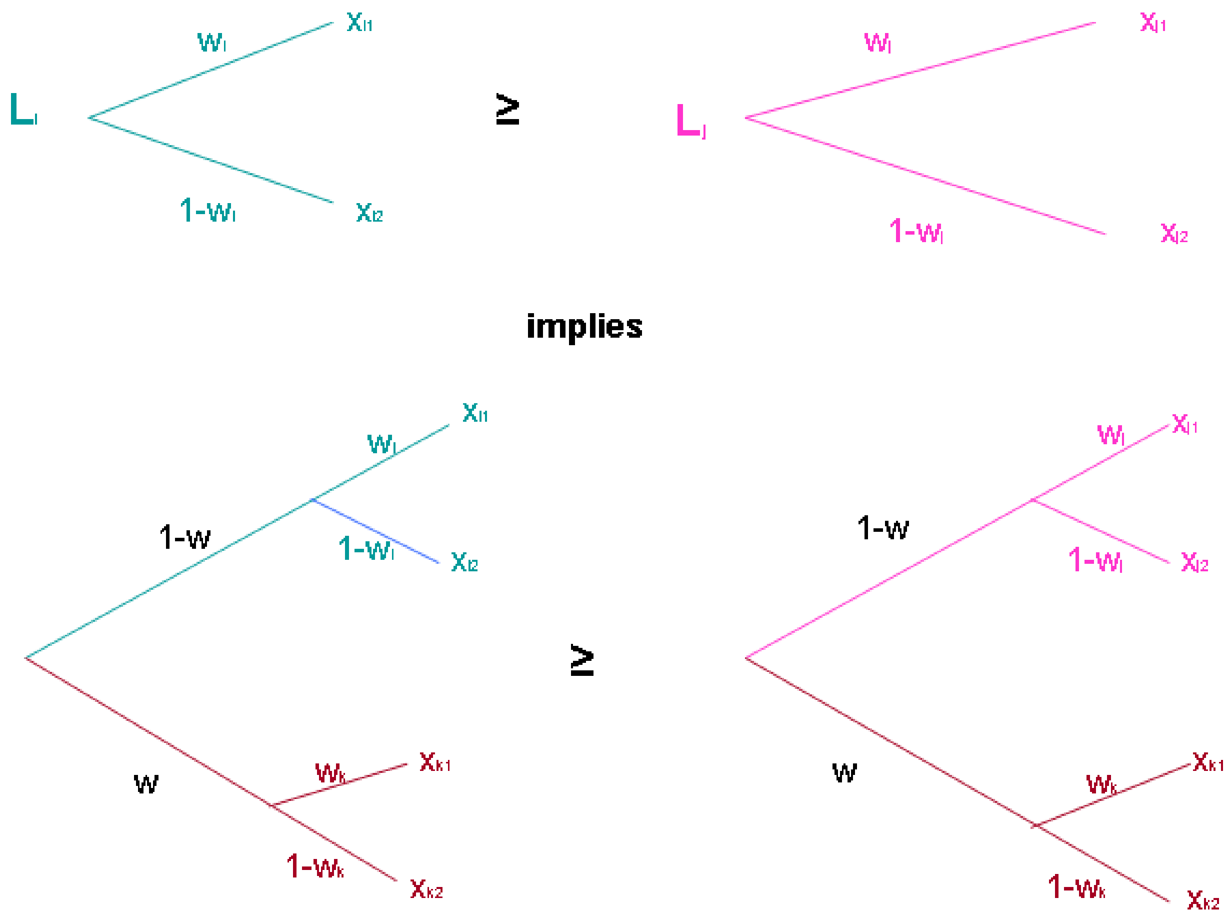

Figure 1: The Independence Axiom

2.2 The vNM-Theorem

Definition A vNM-utility function is a function $U(L_i)$ such that $U(L_i) = \sum_j p_{ij} u(x_{ij})$ where

$L_i \in L$

p_{ij} is the probability of payoff $x_{ij} \in X_i$

$u(x_{ij})$ is given by axiom (c)

Theorem 2.1 *Given axioms (1)-(4), agent will be acting as though she were maximizing a vNM-utility function.*

Proof We'll give proof for the simplest of cases, where there is a lottery L with only two possible outcomes. (However, proof is perfectly analogous for any bounded set of possible outcomes, as you can easily verify.)

Let there be $L = (1 - w, w; x_1, x_2)$

What we're after: U(L) s.t.
$L \sim (1 - U(L), U(L); x_{min}, x_{max})$
Axiom (c): $x_1 \sim (1 - u(x_1), u(x_1); x_{min}, x_{max}) \equiv I(x_1)$
Axiom (d): $L \sim (1 - w, w, ; I(x_1), x_2)$
Axiom (d) again: $L \sim (1 - w, w; I(x_1), I(x_2))$

Note that this is a standard lottery.

Add up the probs:

x_{max}: $Prob(x_{max}) = w \cdot u(x_2) + (1 - w) \cdot u(x_1)$
x_{min}: $Prob(x_{min}) = w \cdot (1 - u(x_2)) + (1 - w) \cdot (1 - u(x_1))$
$= 1 - [w \cdot u(x_2) - (1 - w) \cdot u(x_1)]$

Thus: $L \sim (1 - U(L), U(L); x_{min}, x_{max})$
$\Longrightarrow U(L) = w \cdot u(x_2) + (1 - w) \cdot u(x_1)$
Q.E.D.

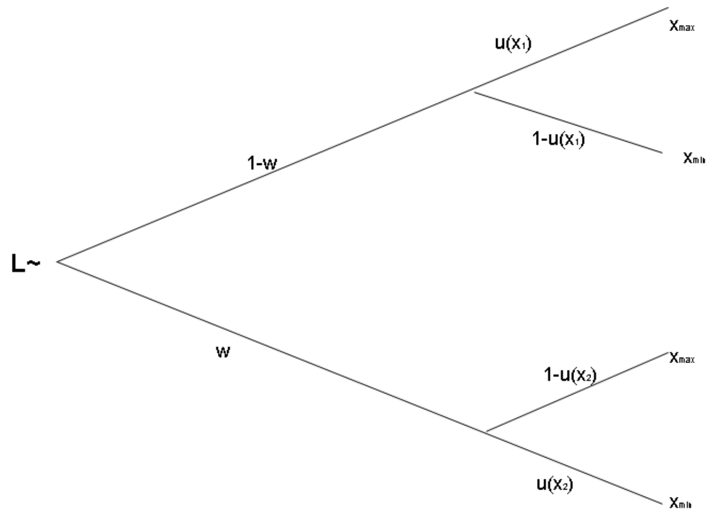

Figure 2: Event Tree

2.3 Basic Properties of vNM-Utility

2.3.1 Transformations

1. $u(x)$ is unique up to a positive *linear* transformation; $u(x)$ (Bernoulli utility) is a *cardinal* utility function (as opposed to the utility functions we know from standard consumer theory, that are *ordinal* utility functions, i.e. unique up to a positive *monotonic* transformation, be that linear or not).

Corollary 2.2 *If u and v be Bernoulli utility functions that represent the same preferences, then there exist constants a, b, with $a \in \mathbb{R}$ and $b \in \mathbb{R}_+$ such that $v(x) = a + bu(x)$.*

Proof Choose a and b such that the following hold:

$$v(x_{max}) = a + bu(x_{max})$$

$$v(x_{min}) = a + bu(x_{min})$$

Now consider $x_{min} < x < x_{max}$. On account of axiom (c) $\exists p \in]0; 1[$ such that

$$x \sim (1-p, p; x_{min}, x_{max})$$

$$\Rightarrow u(x) = pu(x_{max}) + (1-p)u(x_{min})$$

and $v(x) = pv(x_{max}) + (1-p)v(x_{min})$

Plugging in from above, we get:

$$v(x) = p[a + bu(x_{max})] + (1-p)[a + bu(x_{min})]$$

$$\iff v(x) = a + b[pu(x_{max}) + (1-p)u(x_{min})]$$

However, since $pu(x_{max}) + (1-p)u(x_{min}) = u(x)$, it follows q.e.d.

2. U(L) is an *ordinal* utility function, and hence unique up to a positive *monotonic* transformation:

   e.g. $U(L) = \Sigma p_i u(x_i)$
   $V(L) = exp[\Sigma p_i u(x_i)]$

2.3.2 Additional Assumptions

In our analyses, we shall most of the time make some additional assumptions. But, first, we will need some definitions:

Definition A function $f : S_f \longrightarrow A \subset \mathbb{R}$ is said to be (globally) concave iff $\forall (x_1, x_2) \in S_f \times S_f : f[kx_1 + (1-k)x_2] \geq kf(x_1) + (1-k)f(x_2)$ $\forall k \in [0;1]$.

It is said to be strictly concave iff the inequality is strict, i.e. iff $\forall (x_1, x_2) \in S_f \times S_f$: $f[kx_1 + (1-k)x_2] > kf(x_1) + (1-k)f(x_2)$ $\forall k \in]0;1[$.

For at least three times continuously differentiable f, f is strictly concave iff $f''(x) < 0$ $\forall x \in S_f$.

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For at least three times continuously differentiable f, f is strictly convex iff $f''(x) > 0$ $\forall x \in S_f$.

Definition A function $f : S_f \longrightarrow A \subset \mathbb{R}$ is said to be (globally) linear iff it is continuously differentiable and $f''(x) = 0 \; \forall x \in S_f$.

Definition An agent with utility function u is said to be risk–averse iff she prefers the expected value of a lottery L over the lottery itself, i.e. iff

$$E[u(L)] < u[E(L)]$$

Definition An agent with utility function u is said to be risk–attracted or risk–loving iff she prefers a lottery L over $E(L)$, i.e. iff

$$E[u(L)] > u[E(L)]$$

Definition An agent with utility function u is said to be risk–neutral iff she is indifferent between a lottery L and its expected value, i.e. iff

$$E[u(L)] = u[E(L)]$$

In our analyses, we shall usually assume the following:

1. u(x) is differentiable with $u'(x) > 0$

2. u(x) is concave: $u''(x) < 0$

Corollary 2.3 *Let u be a utility function that is at least three times continuously differentiable, with $u' > 0$ and $u'' < 0$.*

Then u depicts risk-aversion.

Proof From Jensen's Inequality (which we shall prove in class), we know that for any concave function u, the following holds:

$$E[u(x)] \leq u(E[x])$$

Q.E.D.

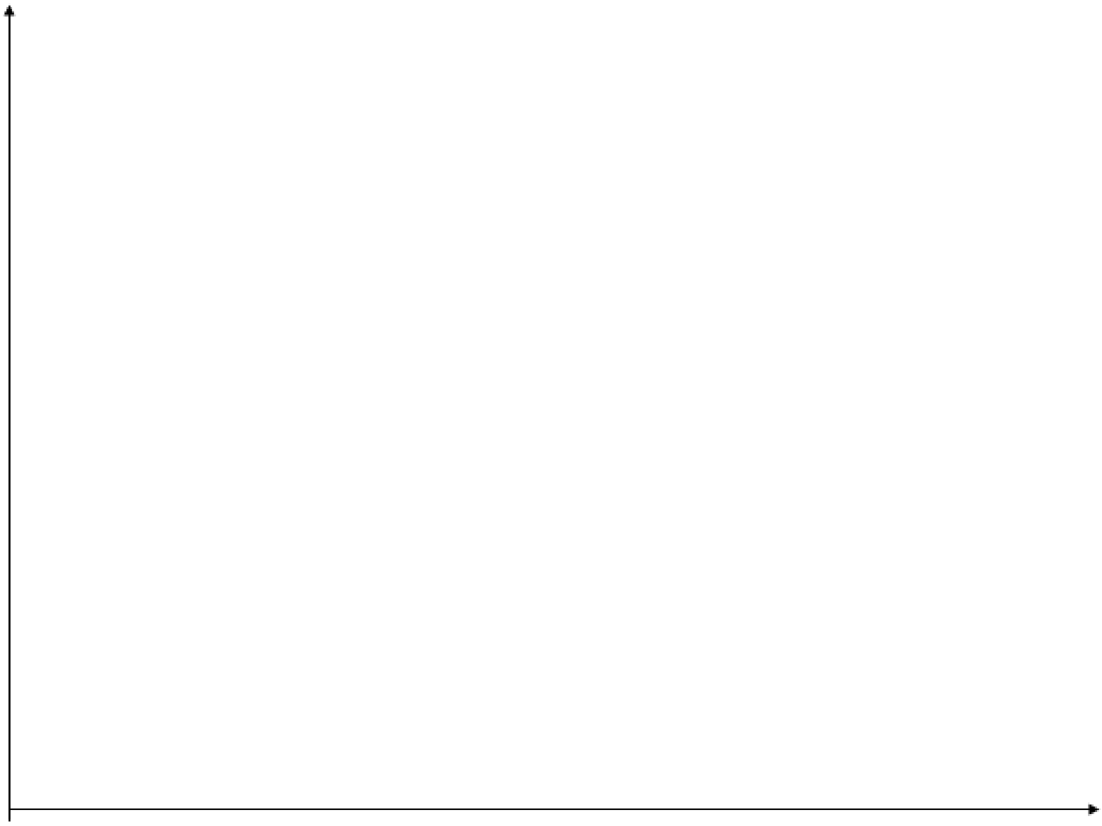

Figure 3: Jensen's Inequality

Example Consider a risk–averse agent with utility function u and a very simple lottery L with $L \sim (1-p, p; x_1, x_2)$

$U(x_1, x_2) = (1-p)u(x_1) + pu(x_2)$
On an indifference curve, the following will hold: $dU = (1-p)u'(x_1)dx_1 + pu'(x_2)dx_2 \overset{!}{=} 0$
$\Longrightarrow -dx_2/dx_1 = \frac{(1-p)u'(x_1)}{pu'(x_2)}$ =:MRS

The Marginal Rate of Substitution (MRS) indicates the rate at which an agent is willing to exchange income in state 2 for income in state 1, with her utility level held constant. It is equal to the absolute value of the slope of her indifference curve at the pertaining point.

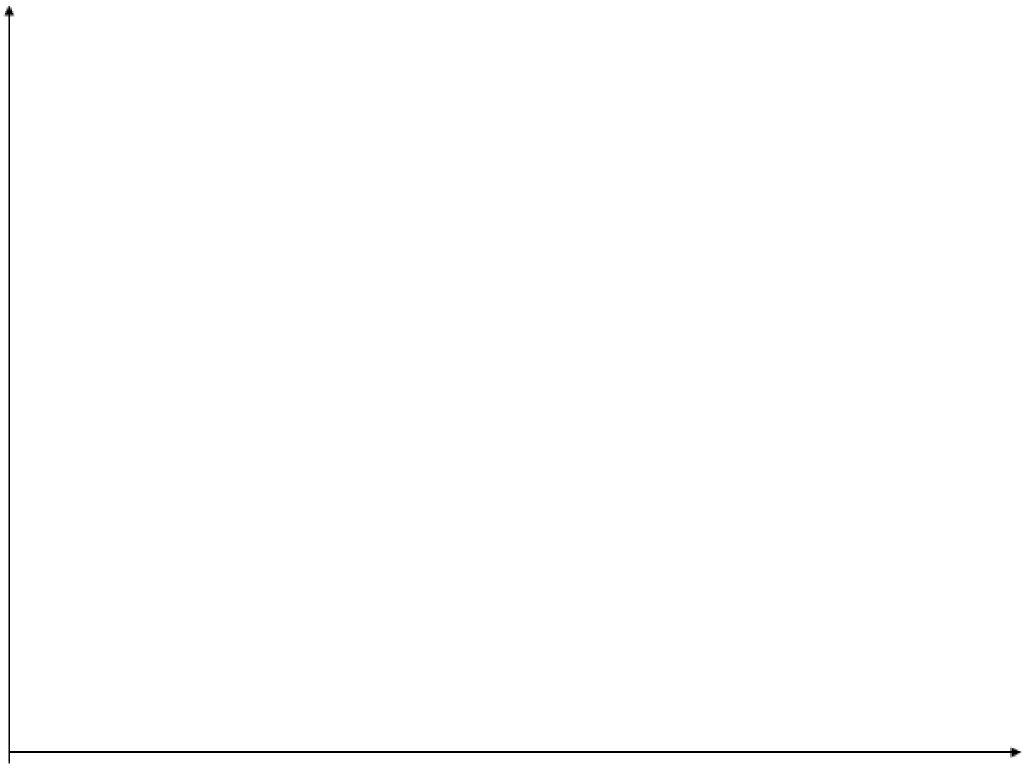

Figure 4: States of the World Diagram

1. $d^2x_2/dx_1^2 = -(1-p)/pu_2'[u_1'' + 1 - p/p(u_1'/u_2')^2 u_2''] > 0$
   i.e. indifference curves are strictly convex

2. $x_1 = x_2 \Longrightarrow dx_2/dx_1 = -(1-p)/p$

   i.e. For variations on the margin at the certainty level, any and every risk–averse agent will behave as though she were risk–neutral. At the certainty level, risk–costs are second—order!

3. Risk–neutral agent (expected value criterion)
   $\Longrightarrow u(x) = x$
   $U = (1-p)x_1 + px_2$
   $\Longrightarrow dx_2/dx_1 = -(1-p)/p$
   i.e. A risk–neutral agent's indifference curves will be linear.

4. Maximin-criterion (Leontief–preferences): $U = min(x_1, x_2)$

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3 Measures of Risk–Aversion

In this chapter we'll examine the question of how to measure risk–aversion, and, hence, how to compare different agents with respect to the degree of their aversion to risk. After some introductory definitions and remarks (3.1.), we shall define the Pratt–Arrow–measures of risk–aversion (3.2.). These being but local measures, we'll move on to try and define risk–attitudes globally (Pratt–theorem, 1964; 3.3.). Finally, we'll briefly look at some alternative measures (3.4.). We'll limit our analysis to lotteries that generate monetary pay–offs (or, equivalently, that generate outcomes that can be translated into monetary pay–offs).

3.1 Introduction

Thus far, we have derived the Pratt–Arrow–theorem and we have defined the concept of risk–aversion. Now, we're interested in determining under what circumstances an agent can be said to be *more* risk–averse than another. In that quest, the following definitions will prove useful:

Definition The *certainty equivalent* $\hat{x}$ of a lottery L is a certain payment of such amount as will make agent indifferent between getting the lottery L and geting its certainty equivalent $\hat{x}$, i.e. (if u be agent's utility function)

$$u(\hat{x}) = E[u(L)]$$

Definition The (equivalent) risk premium r for a *given agent* and a given lottery is defined as

$$r := \bar{x} - \hat{x}$$

where $\bar{x} \equiv E(L)$

Thus, the equivalent risk premium indicates how much of certain income a *given individual* is willing to sacrifice so as to avoid a *given risk*.[2] Hence r amounts to agent's maximum

[2] By contrast, the compensating risk premium is the amount of certain income necessary to make agent willing to take on a given risk. In this course, we shall only use the equivalent risk premium, which will heretofore be plainly referred to as the risk premium r.

willingness to pay for avoidance of a given risk. This implies the following:

Agent A is said to be more risk–averse than agent B if for the same given risk A's willingness to pay to avoid that risk (her risk premium r_A) is larger than B's willingness to pay to avoid that risk (her risk premium r_B).

Corollary 3.1 *(a) The risk premium is strictly positive iff agent is risk–averse.*

(b) The risk premium is strictly negative iff agent is risk–loving.

(c) The risk premium is zero iff agent is risk–neutral.

Proof By definition: $r = \bar{x} - \hat{x}$ (1).

Also by definition: $u(\hat{x}) = E[u(L)]$ (2).

From Jensen's Inequality, we know that $E[u(L)] < u(E[L])$ iff u is strictly concave (3).

From (1) it follows that $r > 0 \iff \bar{x} > \hat{x} \iff u(\bar{x}) > u(\hat{x})$ (since $u' > 0$ by hypothesis).

From (2) it follows that $u(\bar{x}) > u(\hat{x}) \iff u(\bar{x}) > E[u(L)] \iff u(E[L]) > E[u(L)]$. (as $\bar{x} \equiv E[L]$)

From (3) it then follows that $r > 0 \iff u$ strictly concave.

By the same token, (b) and (c) are in a similar way implied by the very definitions of strict convexity and linearity respectively (cf. Chapter 2). ∎

3.2 The Pratt–Arrow–Measures of Risk–Aversion

3.2.1 Introduction

By considering the following figure, the intuition may arise that the size of the risk premium might have something to do with "how strongly convex" the utility function is. Since the second–order derivative is a measure of the curvature of a differentiable function, one might be led to believe that the risk premium was the bigger, i.e. that agent was the more risk–averse, the larger the absolute value of the second–order derivative of her utility function was at the pertaining point. Though not completely mistaken, this idea is, however, abstracting from the fact that our cardinal utility functions are unique but *up to a positive linear transformation* (cf. Chapter 2).

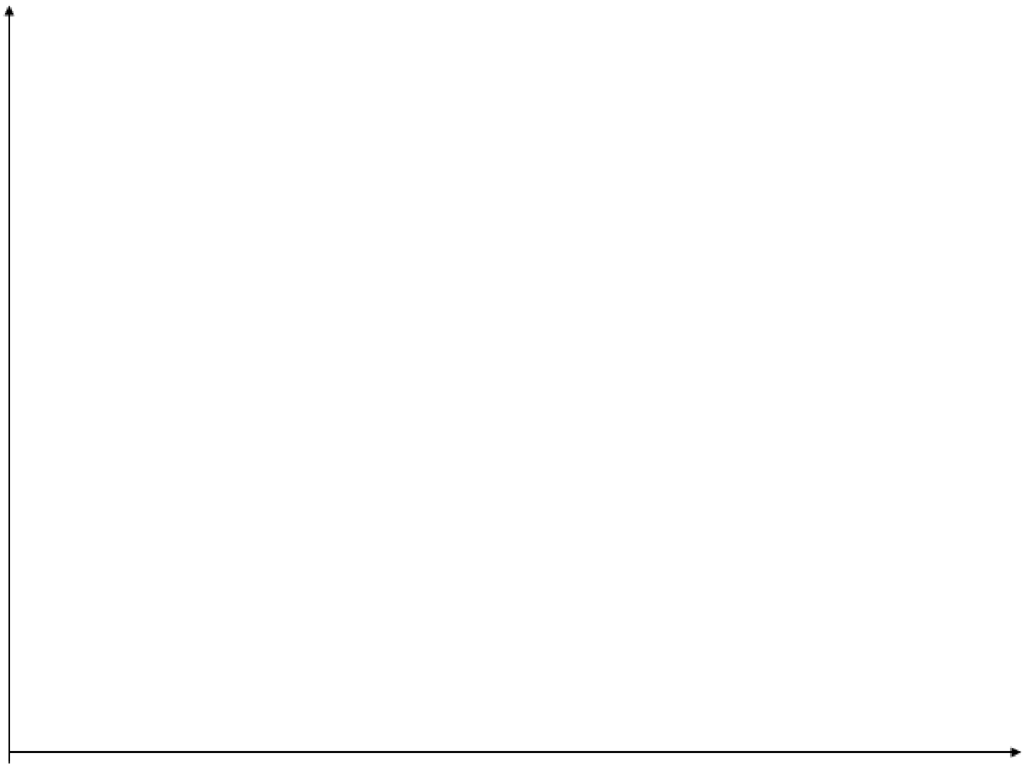

Figure 5: Certainty Equivalent and Risk Premium

Indeed, let us consider two utility functions $u(x)$ and $v(x) = \alpha + \beta u(x)$, with $u(\hat{x}) = E[u(x)]$, i.e. $\hat{x}$ be the certainty equivalent pertaining to utility function u for a given lottery L with realizations x. Then: $E[v(x)] = \alpha + \beta E[u(x)] = \alpha + \beta u(\hat{x}) = v(\hat{x})$, i.e. $\hat{x}$ is the certainty equivalent for lottery L also for utility function v. $E[L] = \bar{x}$ being independent of the utility function, it follows that for a given lottery L, respective risk premia are equal for both u and v. However, $v''(x) = \beta u''(x) \neq u''(x)$ if $\beta \neq 1$.

So, even though risk premia be the same for both utility functions, still, in the generic case, $u''(x) \neq v''(x)$. However, how about $\frac{v''(x)}{v'(x)} = \frac{\beta u''(x)}{\beta u'(x)} = \frac{u''(x)}{u'(x)}$? As a matter of fact $-\frac{u''}{u'}$ is known as the *Pratt–Arrow–coefficient of absolute risk–aversion*, as we shall be examining more closely in the following sub—section.

3.2.2 The Pratt–Arrow–Coefficient of Absolute Risk–Aversion

Hypotheses:

- $u(x)$ be at least twice continously differentiable
- Lottery L with realizations $x_1, \cdots, x_n$
- $\bar{x} = 0$
- $x_1, \cdots, x_n$ be "small" (so we can use a Taylor–Approximation)
- $\sigma_x^3, \sigma_x^4, \cdots$ be "small" (so a *second–order* Taylor–Approximation is good enough)[For standard distributions, this will usually be verified.]
- Initial wealth w

By definition: $u(w + \hat{x}) = Eu(w + x) = \sum p_i u(w + x_i)$

Second Order Taylor Approximation: $u(w + x_i) \approx u(w) + x_i u'(w) + \frac{x_i^2}{2} u''(w)$

Applying the E—operator: $Eu(w + x) \approx u(w) + \frac{\sigma_x^2}{2} u''(w)$

By the same token, $u(w + \hat{x}) = u(w - r_x) \approx u(w) - r_x u'(w)$

Since $u(w + \hat{x}) = Eu(w + x)$, it follows from the above that

$$u(w) - r_x u'(w) \approx u(w) + \frac{\sigma_x^2}{2} u''(w)$$

$$\Longleftrightarrow \; r_x \approx -\frac{u''(w)}{u'(w)} \frac{\sigma_x^2}{2}$$

As we have already mentioned above, $A(w) \equiv -\frac{u''(w)}{u'(w)}$ is known as the Pratt–Arrow–coefficient of absolute risk–aversion. Please do be mindful of the fact that the Pratt–Arrow–coefficent is but a *local* measure, which evaluates risk–aversion for a *given wealth level* w. With larger lotteries, it is but an approximation.

Risk tolerance $R(w)$ is defined as $R(w) = \frac{1}{A(w)}$.

3.2.3 The Pratt–Arrow–Coefficient of Relative Risk–Aversion

Now, consider a lottery of the following multiplicative structure: Final wealth w_i be defined as $w_i \equiv w(1 + x_i)$, with x_i denoting e.g. the interest rate. Whereas, before, the "amount of risk" borne by agent was, as it were, fixed, here, it is increasing in the amount invested (i.e. in initial wealth w).

Calculations and assumptions are analogous to those above:

$$u[w(1+\hat{x})] = Eu[w(1+x)] = \sum p_i u[w(1+x_i)]$$

$$u[w(1+x_i)] \approx u(w) + wx_i u'(w) + \frac{1}{2} w^2 x_i^2 u''(w)$$

$$\Rightarrow Eu[w(1+x)] \approx u(w) + \frac{w^2}{2} \sigma_x^2 u''(w)$$

By the same token, $u[w(1+\hat{x})] = u[w(1-\rho_x)] \approx u(w) - w\rho_x u'(w)$

Since $Eu[w(1+x_i)] = u[w(1+\hat{x})]$, it follows from the above that

$$u(w) + \frac{w^2}{2} \sigma_x^2 u''(w) \approx u(w) - w\rho_x u'(w)$$

$$\iff \rho_x \approx -w \frac{u''(w)}{u'(w)} \frac{\sigma_x^2}{2}$$

Analogously to above, we define $R(w) \equiv -w\frac{u''(w)}{u'(w)}$ as the *Pratt–Arrow–Coefficient of Relative Risk–Aversion* and ρ_x as the relative risk premium. Note that $R(w) = wA(w)$.

3.2.4 The Pratt–Arrow–Coefficient of Partial Risk–Aversion

Consider a lottery of the following structure: $w_i \equiv w_0 + w_1(1 + x_i)$.

Calculations and assumptions are again analogous to those above:

$$u[w_0 + w_1(1+\hat{x})] = Eu[w_0 + w_1(1+x)] = \sum p_i u[w_0 + w_1(1+x_i)]$$

$$u[w_0 + w_1(1+x_i)] \approx u(w_0 + w_1) + w_1 x_i u'(w_0 + w_1) + \frac{1}{2} w_1^2 x_i^2 u''(w_0 w_1)$$

$$\Rightarrow Eu[w_0 + w_1(1+x)] \approx u(w_0 + w_1(1+x)] \approx u(w_0 + w_1) + \frac{1}{2} \sigma_x^2 w_1^2 u''(w_0 + w_1)$$

By the same token, $u[w_0 + w_1(1+\hat{x})] = u[w_0 + w_1(1-\hat{\varrho}_x)] \approx u(w_0 + w_1) - w_1 \hat{\varrho}_x u'(w_0 + w_1)$

Since $Eu[w_0 + w_1(1+x)] = u[w_0 + w_1(1+\hat{x})]$, it follows from the above that

$$u(w_0 + w_1) + \frac{1}{2}\sigma_x^2 w_1^2 u''(w_0 + w_1) \approx u(w_0 + w_1) - w_1 \hat{\varrho}_x u'(w_0 + w_1)$$

$$\iff \hat{\varrho}_x \approx -w_1 \frac{u''(w_0 + w_1)}{u'(w_0 + w_1)} \frac{\sigma_x^2}{2}$$

Again, we define $R_p(w) \equiv -w_1 \frac{u''(w_0+w_1)}{u'(w_0+w_1)}$ as the *Pratt–Arrow–Coefficient of Partial Risk–Aversion* and $\hat{\varrho}_x$ as the partial risk premium. Note that $R_p(w_0 + w_1) = w_1 A(w_0 + w_1) = \frac{w_1}{w_0+w_1} R(w_0 + w_1)$.

3.2.5 Some Common Assumptions

First, we need some definitions:

Definition An at least twice continuously differentiable utility function u is said to exhibit the property of decreasing absolute risk aversion (DARA) iff $\frac{dA(w)}{dw} < 0$.

It is said to exhibit the property of constant absolute risk aversion (CARA) iff $\frac{dA(w)}{dw} = 0$.

It is said to exhibit the property of increasing absolute risk aversion (IARA) iff $\frac{dA(w)}{dw} > 0$.

Definition An at least twice continuously differentiable utility function u is said to exhibit the property of decreasing relative risk aversion (DRRA) iff $\frac{dR(w)}{dw} < 0$.

It is said to exhibit the property of constant relative risk aversion (CRRA) iff $\frac{dR(w)}{dw} = 0$.

It is said to exhibit the property of increasing relative risk aversion (IRRA) iff $\frac{dR(w)}{dw} > 0$.

The following assumptions are usually held to be plausible:

- $\frac{dA(w)}{dw} < 0$ i.e. decreasing absolute risk aversion (DARA) is implied, meaning, intuitively, that a pauper should suffer more from having to bet 10$ than Bill Gates.
- It is often assumed that $\frac{dR(w)}{dw} \geq 0$ i.e. constant (CRRA) or increasing (IRRA) relative risk–aversion is implied.

Remember that $R(w) = wA(w)$, which implies that $\frac{dR(w)}{dw} = A(w) + w\frac{dA(w)}{dw}$.

Here, the term $A(w)$ (which, for a risk–averse agent, will always be > 0) denotes that for a lottery of the structure $w(1+x)$, more income will be risky as w increases. This substitution effect will lead to an increasing of risk aversion as initial wealth w rises. However, for DARA utility ($\frac{dA(w)}{dw} < 0$), the income effect will work in the opposite direction, i.e. increasing initial wealth will lead to a tapering of risk aversion as DARA—individuals will wax less sensitive to their income being risky as they grow wealthier.

Example Be $x_i \in [-0,1;0,1]$. For $w = 100$, $w_i \in [90;110]$. However, for $w = 10.000$, $w_i \in [9.000;11.000]$.

3.2.6 Some Utility Functions

1. Quadratic Utility

$$u(w) = w - \alpha w^2$$

As will be discussed in class, quadratic functions are meaningful utility functions only for a limited support. None the less, such functions are often used on account of their being very easy to handle as agent is exclusively interested in the expected value and the variance of a distribution (cf. class). For values of w within the support, $A(w)$ is given by:

$$A(w) = -\frac{-2\alpha}{1-2\alpha w}$$

It follows that, as we would be wishing for from a decent utility function, $A(w) > 0$ within function u's support. However:

$$\frac{dA(w)}{dw} = (\frac{2\alpha}{1-2\alpha w})^2 > 0$$

i.e. IARA is implied.

2. Logarithmic Utility

$$u(w) = \ln w$$

$$A(w) = \frac{w}{w^2} = \frac{1}{w}$$

$$A'(w) = -\frac{1}{w^2} < 0 \qquad \text{DARA}$$

$$R(w) = wA(w) = 1 \qquad \text{CRRA}$$

3. Power Utility

$$u(w) = \frac{w^{1-\sigma}}{1-\sigma}$$

$$u'(w) = w^{-\sigma}$$

$$u''(w) = -\sigma w^{-\sigma-1}$$

$$\Rightarrow A(w) = \frac{\sigma}{w}$$

$$A'(w) = -\frac{\sigma}{w^2} < 0 \qquad \text{DARA}$$

$$R(w) = \sigma \qquad \text{CRRA}$$

4. Exponential Utility

$$u(w) = -e^{-\alpha w}$$

$$A(w) = \alpha \qquad \text{CARA}$$

$$R(w) = wA(w) = w\alpha \qquad \text{IRRA (for } \alpha > 0)$$

5. Hyperbolic Utility (HARA—functions)

$$u(w) = \frac{1-\gamma}{\gamma}\left(\frac{\alpha w}{1-\gamma} + \beta\right)^{\gamma} \text{ with} \frac{\alpha w}{1-\gamma} + \beta \geq 0$$

$$A(w) = \frac{\alpha}{\frac{\alpha w}{1-\gamma} + \beta}$$

i.e. HARA—functions will exhibit linear risk tolerance (LRT), meaning $A^{-1}(w)$ is linear in w.

3.3 Global Risk–Aversion (Pratt, 1964)

Consider a given lottery L and two utility functions u_A and u_B.

Theorem 3.2 *The following statements are equivalent:*

(a) $A_A(w) \geq A_B(w) \qquad \forall w$

(b) $r_A(x,w) \geq r_B(x,w) \qquad \forall w, x$

(c) $u_A(.) \geq G(u_B(.))$*, with* G *being a concave function*

(d) $\frac{u_A(w_3)-u_A(w_2)}{u_A(w_1)-u_A(w_0)} \leq \frac{u_B(w_3)-u_B(w_2)}{u_B(w_1)-u_B(w_0)}$ $\forall w_0,\ w_1,\ w_2,\ w_3$ *with* $w_0 < w_1 \leq w_2 < w_3$

We'll not give complete proof of the entire theorem; the following examples may suffice:

- (c) $\Rightarrow$ (b)

$$u_A(w + \bar{x} - r_A(x,w))$$
$$= E[u_A(w+x)]$$

$$
\begin{aligned}
&= E[G(u_B(w+x))] \\
&\leq G(E[u_B(w+x)]) \qquad\qquad \text{(Jensen)} \\
&= G(u_B(w+\bar{x}-r_B(x,w))) \\
&= u_A(w+\bar{x}-r_B(x,w))
\end{aligned}
$$

Q.E.D.

- (b) $\Rightarrow$ (c)

$$u_A(w+\bar{x}-r_A) = E[u_A(w+x)] = E[G(u_B(w+x))]$$

$$u_A(w+\bar{x}-r_B) = G(u_B(w+\bar{x}-r_B)) = G(E[u_B(w+x)])$$

From Jensen's Inequality, it follows Q.E.D.

- (c) $\Rightarrow$ (a)

$$
\begin{aligned}
A_A(w) = -\frac{u_A''(w)}{u_A'(w)} &= -\frac{G'u_B'' + G''u_B'^2}{G'u_B'} \\
&= A_B(w) - \frac{G''u_B'}{G'}
\end{aligned}
$$

Since $G'' < 0 < G'$ and $u' > 0$, it follows Q.E.D.

- (a) $\Rightarrow$ (c)

$$A_A(w) = -\frac{u_A''(w)}{u_A'(w)} = -\frac{G'u_B'' + G''u_B'^2}{G'u_B'}$$

For $G' > 0$ and $u_B' > 0$, it follows that $A_A(w) \geq A_B(w) \iff G'' \leq 0$

Q.E.D.

3.4 Extensions

3.4.1 The Measure of Absolute Prudence (Kimball, Econometrica 1990)

Let there be two periods, present and future, with c denoting present consumption and β designating agent's discount factor. Future consumption is subject to uncertainty x, with $\bar{x} = 0$.

$$\max_c \{u(c) + \beta E[u(w-c+x)]\}$$

$$\text{FOC:} \qquad u'(c) \stackrel{!}{=} \beta E[u'(w-c+x)]$$

Let ψ denote the precautionary premium, such that $E[u'(w-c+x)] = u'(w-c-\psi)$. Using first— and second—order Taylor Approximations (as above), we can compute:

$$u'(w-c+x) \approx u'(w-c) + xu''(w-c) + \frac{1}{2}x^2u'''(w-c)$$

$$\Rightarrow Eu'(w-c+x) \approx u'(w-c) + \frac{\sigma_x^2}{2}u'''(w-c)$$

By the same token, $u'(w-c-\psi) \approx u'(w-c) - \psi u''(w-c)$.

Since $Eu'(w-c+x) = u'(w-c-\psi)$, it follows that

$$u'(w-c) + \frac{\sigma_x^2}{2}u'''(w-c) \approx u'(w-c) - \psi u''(w-c)$$

$$\iff \psi \approx -\frac{u'''(w-c)}{u''(w-c)}\frac{\sigma_x^2}{2}$$

$p_x(w-c) \equiv -\frac{u'''(w-c)}{u''(w-c)}$ is defined as the *Measure of Absolute Prudence*, whereas ψ is termed the *precautionary premium*. The higher $p_x(w-c)$, the more agent will save (in absolute terms), and, hence, the less agent will consume in the present (i.e. the lower c). $p_x \uparrow \Rightarrow \psi \uparrow \Rightarrow \beta E[u'(w-c+x)] \uparrow \Rightarrow u'(c) \uparrow \Rightarrow c \downarrow$

Most of times, it will be assumed that $\frac{dp_x(w-c)}{dw} < 0$, i.e. the wealthier agent is the more she will consume in the present.

3.4.2 Some Further Concepts

- Ross, 1981: u_A is more risk–averse than u_B if $\exists \lambda : \frac{u''_A(y)}{u''_B(y)} \geq \lambda \geq \frac{u'_A(y)}{u'_B(y)}$

  This measure is used to ascertain preferences over lotteries in cases where there are background risks (i.e. risks for which there is no market).

- The Measure of Absolute Temperance (Gollier/Pratt, 1996; Kimball, 1992) The Measure of Absolute Temperance $t(w)$ is defined as $t(w) \equiv -\frac{u''''(w)}{u'''(w)}$.

- Standard Risk Aversion (Kimball, 1993)

$$p'_x(w) \leq 0 \text{ or } t(w) \geq p_x(w)$$

$$A'(w) \leq 0 \text{ or } p_x(w) \geq A(w)$$

  etc. etc.

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4 Measures of Risk

4.1 Introduction

Thus far, we have been looking at ways to rank *individuals* with respect to their risk aversion. In this chapter, we'll endeavor to rank monetary *lotteries* with respect to their "riskiness". For, indeed, in this, it is not always good enough just to compare the variances of two lotteries with the same expected value.

Example Let there be two Lotteries A and B, with $A \sim (\frac{7}{8}, \frac{1}{8}; 1, 9)$ and $B \sim (\frac{1}{2}, \frac{1}{2}; 0, 4)$.

$E[A] = 2 = E[B]$; $V[A] = 7$; $V[B] = 4$

Now, let agent's utility be given by $u(w) = \sqrt{w}$.

$E_A[u] = \frac{10}{8}$; $E_B[u] = 1 < \frac{10}{8}$

i.e. agent with utility $u(w) = \sqrt{w}$ will prefer lottery A over lottery B, even though $E[A] = E[B]$ and $V[A] > V[B]$.

As we have already seen in class, however, agent will only be interested in the expected value and the variance of a distribution if she exhibits quadratic utility.

4.2 Stochastic Dominance

Stochastic Dominance is a concept that allows us to rank distributions as to their "riskiness". While *satisfying* the property of *transitivity*, this concept, however, is *not complete*, i.e. it will never be possible to rank all distributions. We'll first take a look at first–order stochastic dominance (FOSD), by which we are able to give a preference ordering for all utility functions u with $u' > 0$ (4.2.2.), before turning to second–order stochastic dominance (SOSD), which gives a preference ordering for all utility functions u with $u'' < 0 < u'$ (4.2.3.). There are concepts of higher–order stochastic dominance, which allow for the ranking of a vaster class distributions, but which, in turn, require starker restrictions on utility function u.

4.2.1 Refresher: Integration by Parts

Lemma 4.1 *Let $u(x)$ and $v(x)$ be two continuously differentiable functions, with $u : [a;b] \longrightarrow \mathbb{R}$, $v : [a;b] \longrightarrow \mathbb{R}$, $(a,b) \in \mathbb{R}^2$. Then, the following equality will hold:*

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,dx$$

Proof As we know from High School: $(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$ Since this holds for all $x \in [a;b]$, it follows that

$$\int_a^b (u(x)v(x))'\,dx = \int_a^b u'(x)v(x)\,dx + \int_a^b u(x)v'(x)\,dx$$
$$\iff \int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,dx$$

Q.E.D.

4.2.2 First–Order Stochstic Dominance (FOSD)

Let there be two distributions A and B, determined by their cumulative distribution functions F_A and F_B, respectively. f_A and f_B denote the respective densities, which shall exist by hypothesis (i.e. we suppose the CDFs to be continuously differentiable).

$$E_B[u] \geq E_A[u] \iff \int_a^b u(x)f_B(x)\,dx \geq \int_a^b u(x)f_A(x)\,dx$$
$$\iff \int_a^b u(x)[f_B(x) - f_A(x)]\,dx \geq 0$$
$$\iff [u(x)(F_B(x) - F_A(x))]_a^b - \int_a^b u'(x)(F_B(x) - F_A(x))\,dx \geq 0$$
$$\iff \int_a^b u'(x)(F_A(x) - F_B(x)) \geq 0$$

For $u' > 0$, this last inequality will always hold if $\forall x \in [a;b] : F_A(x) \geq F_B(x)$. This leads us to the definition of First–Order Stochastic Dominance:

Definition Let $F_A(x)$ and $F_B(x)$ be two continuously differentiable cumulative distribution functions. Then F_B is said to first–order stochastically dominate F_A iff

$$\forall x \in \mathbb{R} : F_A(x) \geq F_B(x)$$

AND

$$\exists x \in \mathbb{R} : F_A(x) > F_B(x)$$

The following theorem is immediately implied by the above:

Theorem 4.2 *Risk–loving, risk–neutral, and risk–averse individuals with a positive marginal utility in income prefer the first–order stochastically dominating distribution of income.*

Corollary 4.3

$$F_B(x) \succ^{FOSD} F_A(x) \Rightarrow E_B[x] > E_A[x]$$

Proof

$$E_i[x] = \int_a^b x f_i(x)\,dx = [xF_i(x)]_a^b - \int_a^b F_i(x)\,dx$$

$$= b - \int_a^b F_i(x)\,dx$$

$$E_B(x) - E_A(x) = \int_a^b F_A(x) - F_B(x)\,dx > 0 \qquad \text{(on account of FOSD)}$$

Q.E.D.

4.2.3 Second–Order Stochastic Dominance (SOSD)

Let there be two distributions A and B, determined by their cumulative distribution functions F_A and F_B, respectively. f_A and f_B denote the respective densities, which shall exist by hypothesis (i.e. we suppose the CDFs to be continuously differentiable).

$$E_B[u] \geq E_A[u] \iff \int_a^b u(x) f_B(x)\,dx \geq \int_a^b u(x) f_A(x)\,dx$$

$$\iff \int_a^b u(x)[f_B(x) - f_A(x)]\,dx \geq 0$$

$$\iff [u(x)(F_B(x) - F_A(x))]_a^b - \int_a^b u'(x)(F_B(x) - F_A(x))\,dx$$

$$\iff \int_a^b u'(x)(F_A(x) - F_B(x)) \geq 0$$

Define $T'(x) \equiv F_B(x) - F_A(x)$. It follows that $T(x) = \int_{-\infty}^{x} F_B(u) - F_A(u)\,du$, and that $T(a) = 0$. Continuing our calculations:

$$\int_a^b u'(x)(F_A(x) - F_B(x)) \geq 0$$

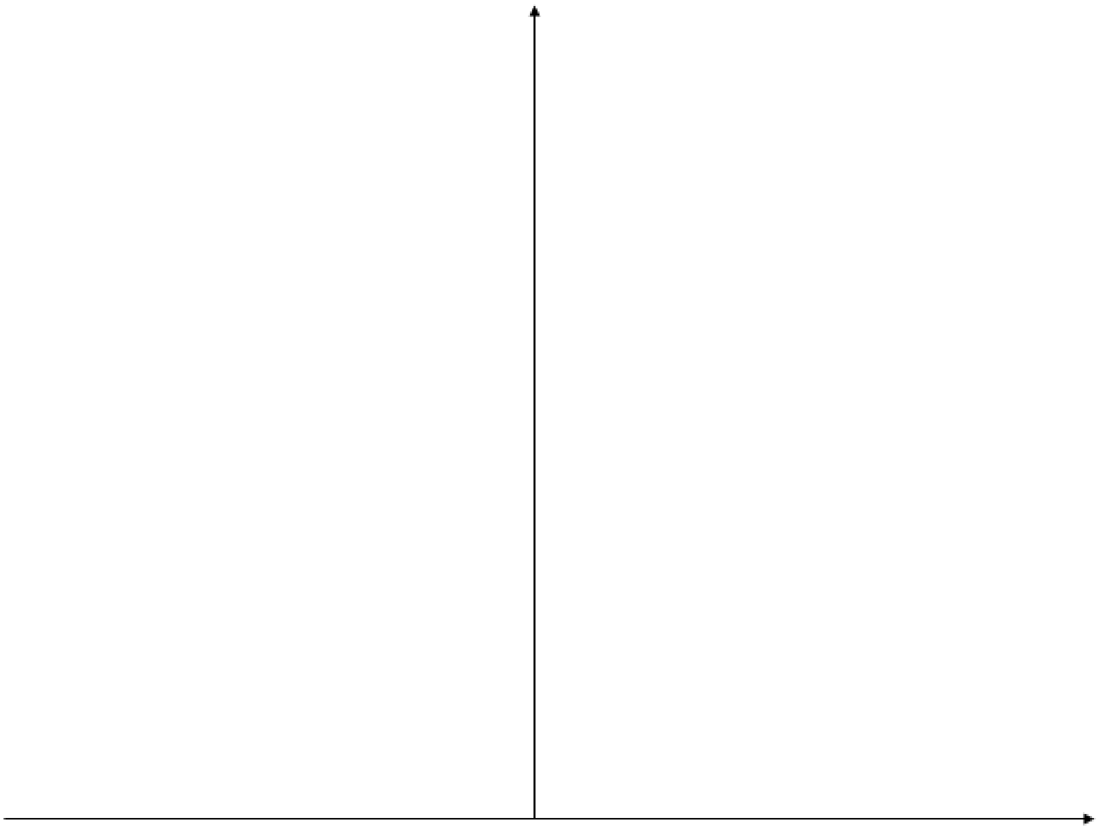

Figure 6: First–Order Stochastic Dominance

$$\iff [-u'(x)T(x)]_a^b + \int_a^b u''(x)T(x)\,dx \geq 0$$

$$\iff -u'(b)T(b) + \int_a^b u''(x)T(x)\,dx \geq 0$$

This inequality will always hold if $\forall x \in [a;b] : T(x) \geq 0$. This leads us to the definition of second–order stochastic dominance:

Definition Let $F_A(x)$ and $F_B(x)$ be two continuously differentiable cumulative distribution functions. Then F_B is said to second–order stochastically dominate F_A iff

$$\forall x \in [a;b] : \int_a^b F_B(x)\,dx \leq \int_a^b F_A(x)\,dx$$

The following theorem immediately follows from the above:

Theorem 4.4 Risk–averse *individuals with a positive marginal utility in income prefer the second–order stochastically dominating distribution of income.*

Please NOTE that whilst FOSD works for all individuals with positive marginal utility of income, SOSD works *ONLY for risk–averse individuals.*

Also NOTE that the ordering of distributions given by FOSD and SOSD is *NOT complete*, i.e. ALL possible income distributions CANNOT be ranked by FOSD or SOSD.

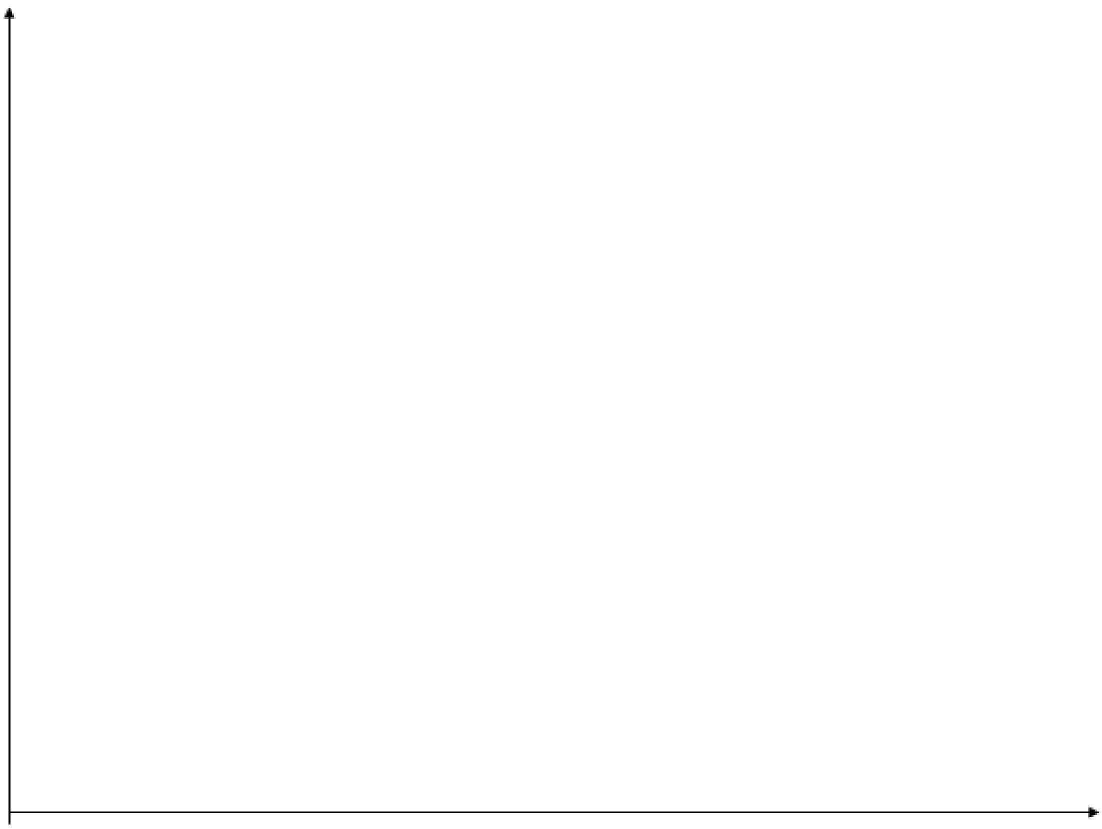

Figure 7: Second–Order Stochastic Dominance I

NOTE that the area above the cumulative distribution function (CDF) is equal to the expected value of the distribution.

Proof

$$\bar{x} = \int_a^b x f(x)\, dx$$

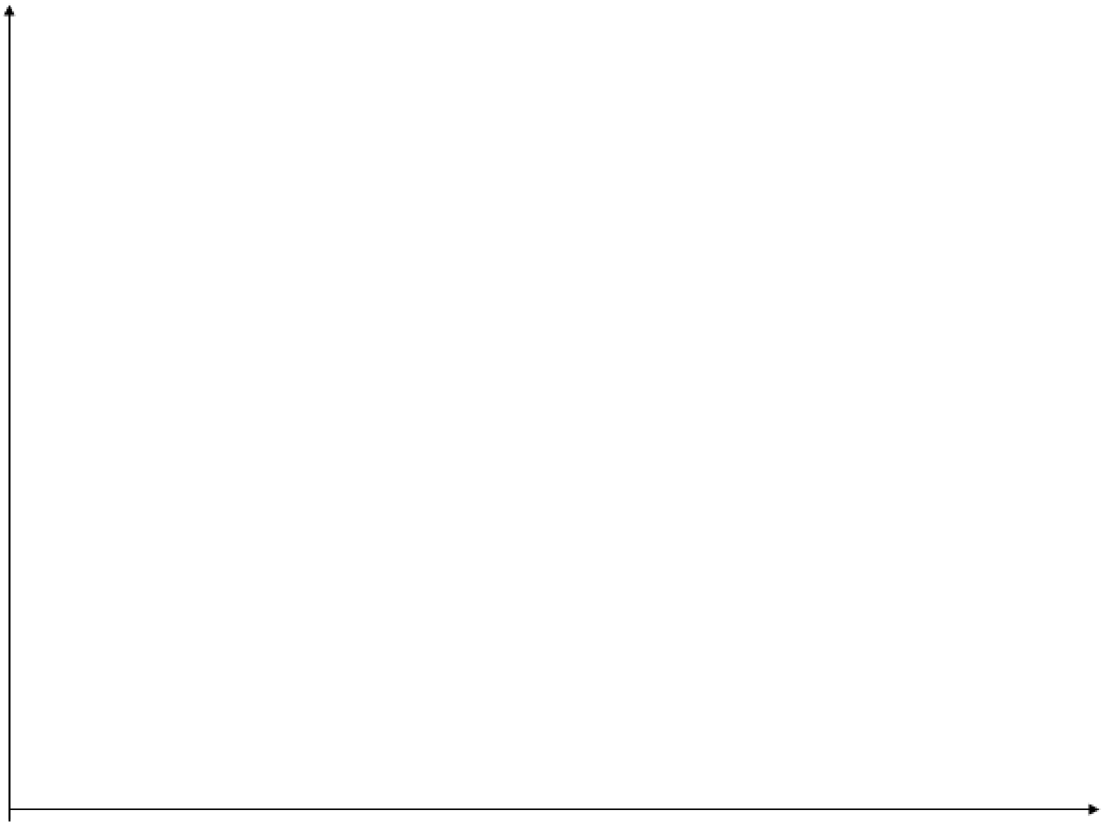

Figure 8: First–Order Stochastic Dominance II

$$= [x(F(x) - 1)]_a^b - \int_a^b (F(x) - 1)\, dx$$

$$= a + \int_a^b (1 - F(x))\, dx$$

Q.E.D.

Definition $F_A(x)$ is said to be a mean–preserving spread (MPS) of $F_B(x)$ iff $F_B \succ^{SOSD} F_A$ AND $E_A(x) = E_B(x)$.

Definition A distribution A with a cumulative distribution function $F_A :\ S_A \longrightarrow \mathbb{R}$ is said to be a *strong increase in risk* (Meyer & Ormiston) of another distribution B with cumulative distribution function $F_B :\ S_B \longrightarrow \mathbb{R}$ iff

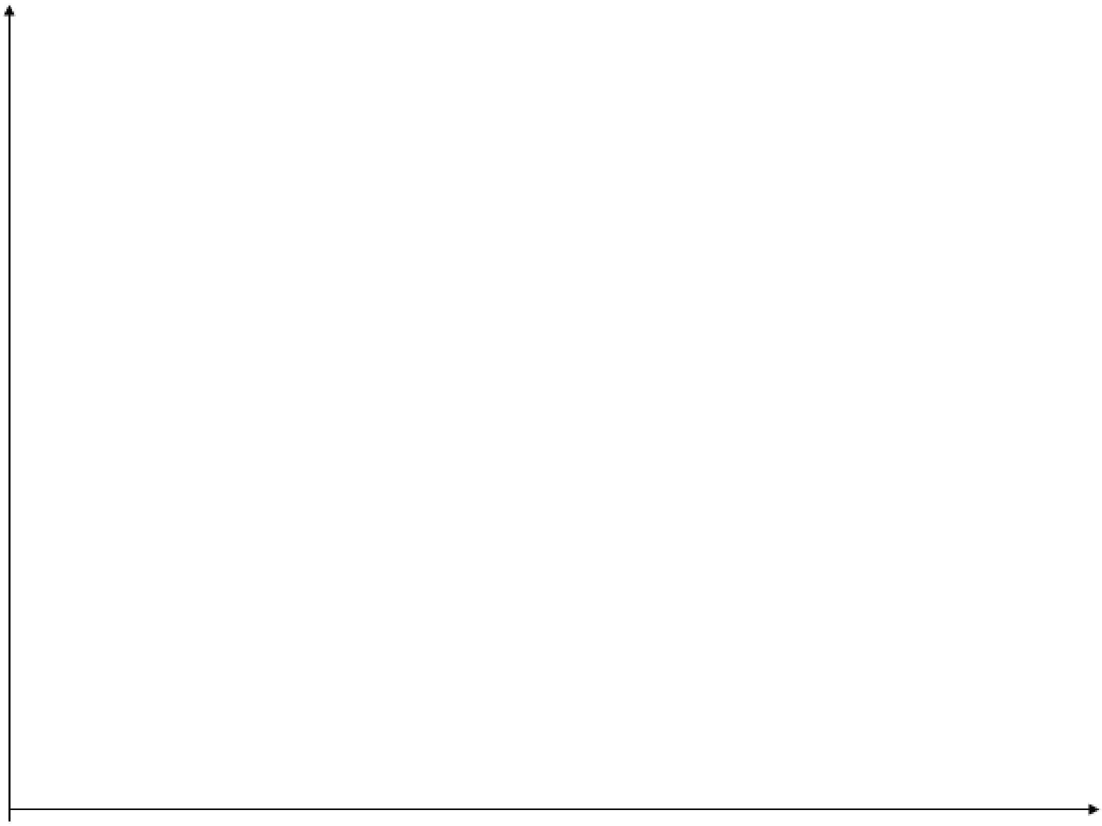

Figure 9: Mean–Preserving Spread

(i) A is a mean–preserving spread of B AND

(ii) $S_A \setminus S_B \neq \emptyset$

4.3 The Rothschild–Stiglitz Theorem, 1970

Theorem 4.5 *Let there be two lotteries over* $x \in [a; b]$*, A and B, with* $E_A(x) = E_B(x)$*.*

The following statements are equivalent:

(a) Any and every *risk–averse agent will prefer lottery B over lottery A*

(b) $\forall x \in [a; b] : \int_a^x (F_B(u) - F_A(u))\, du \geq 0$

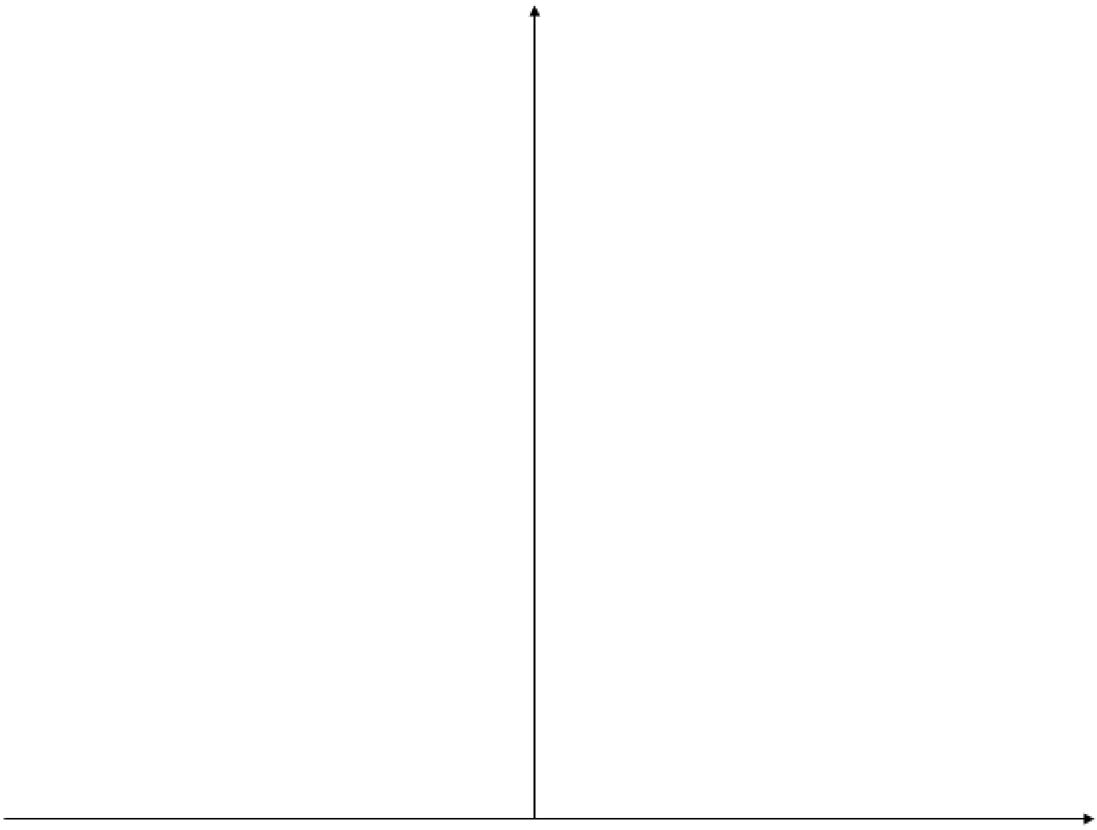

Figure 10: Mean–Preserving Spread II

(c) A is a MPS of B

(d) A is equal to B but for addition of white noise

That (b) $\Rightarrow$ (a) we have already seen above; we'll not prove (a) $\Rightarrow$ (b) here. (b) $\iff$ (c) is true by the very definition of a MPS. Now, we shall prove (d) $\Rightarrow$ (a):

Proof Let lottery A be defined over $y \in [a; b]$, whilst lottery B is defined over $x \in [a; b]$.

White noise ϵ: $y = x + \epsilon$, with $E[\epsilon \mid x] = 0$

(α) Show that both distributions have the same mean

$$E_y[y] = E_{x,\epsilon}[x + \epsilon] = E_x[E_\epsilon[(x + \epsilon) \mid x]] = E_x[x]$$

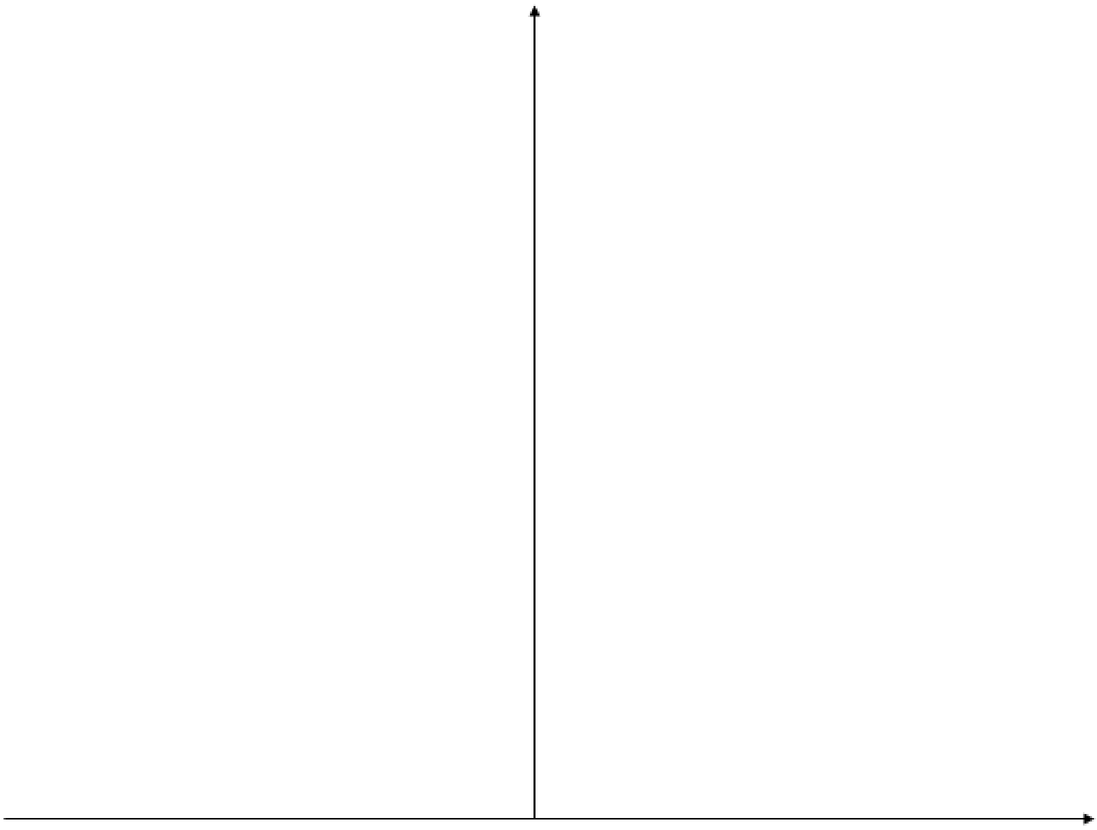

Figure 11: Mean–Preserving Spread III

Q.E.D.

(β) Show that any risk–averse individual would prefer B over A

$$
\begin{aligned}
E_y[u(y)] &= E_{x,\epsilon}[u(x+\epsilon)] \\
&= E_x[E_\epsilon[u(x+\epsilon) \mid x]] \\
&< E_x[u(x + E_\epsilon[\epsilon \mid x])] \qquad \text{(Jensen)} \\
&= E_x[u(x)]
\end{aligned}
$$

Q.E.D.

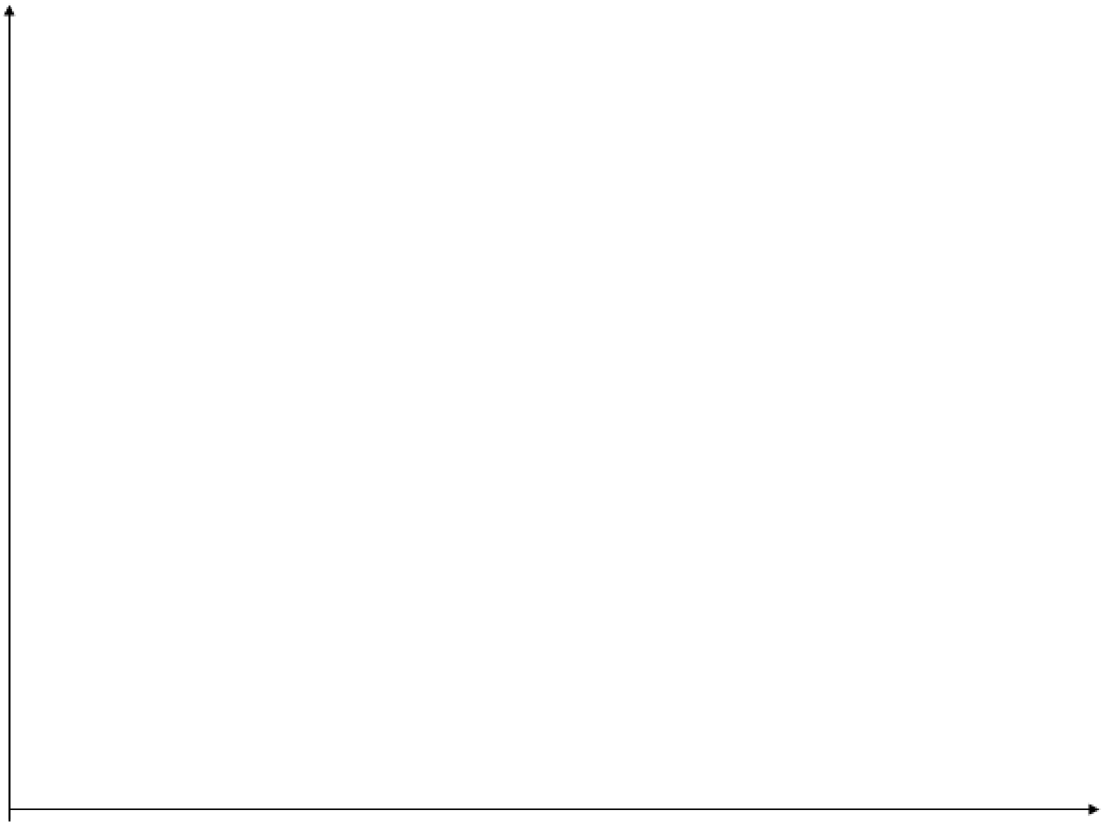

Figure 12: Strong Increase in Risk

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5 Applications

5.1 Optimum Portfolio Selection

5.1.1 The Basic Problem

Agent can invest her initial wealth w_0 either buying a riskless asset that pays out $(1+i)$ with certainty or purchasing a risky asset with payout $(1+\tilde{x})$ (where $\mu \equiv E[\tilde{x}]$). Let m

denote the amount invested in the risk–free asset and a the sum invested in the risky asset. Thus, final wealth $\tilde{w}$ is given by $\tilde{w} = m(1+i) + a(1+\tilde{x})$.

$$\max_{a,m} E[u(m(1+i) + a(1+\tilde{x}))] \qquad \text{s.t. } m + a \leq w_0$$

Since the constraint binds, this is tantamount to:

$$\max_{a} E[u(w_0(1+i) + a(\tilde{x} - i)]$$

$$\text{FOC:} \qquad \frac{\partial Eu}{\partial a} = E[u'(w_0(1+i) + a(\tilde{x}-i))(\tilde{x}-i)] \overset{!}{=} 0$$

$$\text{SOC:} \qquad \frac{\partial^2 Eu}{\partial a^2} = E[u''(.)(\tilde{x}-i)^2] < 0$$

SOC is fulfilled for risk–averse agents (i.e. for $u'' < 0$).

$$\frac{\partial Eu}{\partial a}|_{a=0} = E[u'(w_0(1+i))(\tilde{x}-i)]$$

$$= u'(w_0(1+i))E[\tilde{x}-i]$$

$$= u'(w_0(1+i))(\mu - i)$$

IMPORTANT RESULT:

Since $u' > 0$, it follows that $a^* > 0 \iff \mu > i$ and $a^* < 0 \iff \mu < i$, i.e. ANY RISK-AVERSE AGENT WILL (TO SOME DEGREE) PARTAKE IN A RISKY PROJECT IF THE EXPECTED RETURN FROM DOING SO IS STRICTLY POSITIVE. Recall from Chapter 2 that, at the certainty level, risk–costs are second–order.

We shall now turn to comparative static analysis. First, though, as a refresher, we'll briefly recall the Implicit Function Theorem.

5.1.2 Refresher: Implicit Function Theorem (IFT)

Theorem 5.1 *Let $f(x,y)$ be a continuously differentiable function with $f(x,y) = 0$ and $\frac{\partial f}{\partial x}|_{x,y} \neq 0$. Then, the following equality will hold:*

$$\frac{dx}{dy} = -\frac{\frac{\partial f(x,y)}{\partial y}}{\frac{\partial f(x,y)}{\partial x}}$$

5.1.3 Comparative Statics: The Effect of a Change in w_0

Define $M \equiv \frac{\partial Eu}{\partial a} = E[u'(w_0(1+i)+a(\tilde{x}-i))(\tilde{x}-i)] \overset{!}{=} 0$ (FOC). From the IFT, we know:

$$\frac{da^*}{dw_0} = -\frac{\frac{\partial M}{\partial w_0}}{\frac{\partial M}{\partial a}}$$

Since the SOC holds, we know that $\frac{\partial M}{\partial a} < 0$. Hence, it follows that $sgn[\frac{\partial a^*}{\partial w_0}] = sgn[\frac{\partial M}{\partial w_0}]$.

$$\frac{\partial M}{\partial w_0} = E[u''(w_0(1+i)+a(\tilde{x}-i))(\tilde{x}-i)](1+i)$$

Since $u'' < 0$, it can easily be seen that $sgn[\frac{\partial M}{\partial w_0}] = -sgn[\tilde{x}-i]$.

Now recall that $A(y) = -\frac{u''(y)}{u'(y)} \iff -u''(y) = A(y)u'(y)$. Furthermore, define $y_f \equiv w_0(1+i)+a(\tilde{x}-i)$.

$$\Rightarrow \frac{\partial M}{\partial w_0} = E[-A(y_f)u'(y_f)(\tilde{x}-i)](1+i)$$

Recall the FOC: $E[u'(y_f)(\tilde{x}-i)] \overset{!}{=} 0$. So, for CARA, i.e. for $A(y_f) = const \forall y_f$, $\frac{\partial M}{\partial w_0} = 0$. Thus, for CARA utility, initial wealth has no effect whatever on the amount actually invested in a risky project. Now, define $f : \tilde{x} \mapsto A(y_f)u'(y_f)(\tilde{x}-i)$, and consider the following figure:

From the FOC, we know that, for CARA, the area delimited by the graph of $f(\tilde{x})$ which is to the left of $\tilde{x} = i$ (area B) is equal to the area delimited by the graph of $f(\tilde{x})$ which is to the right of $\tilde{x} = i$ (area C). For DARA, however, $A(y_f)$ decreases with $\tilde{x}$, so we get the orange–colored graph. As can easily be seen, the area delimited by the orange–colored graph situated to the left of $\tilde{x} = i$ is larger than area B whereas that situated to the right of $\tilde{x} = i$ is smaller than area C. It follows that, for DARA, $\int_{-\infty}^{+\infty} f(\tilde{x}) < 0$, and, hence, $\frac{\partial M}{\partial w_0} > 0 \Rightarrow \frac{da^*}{dw_0} > 0$, i.e.: For DARA utility, the amount invested in a risky project increases in agent's initial wealth; risky projects are superior goods.

Applying analogous logic to the case of IARA, we get: $\int_{-\infty}^{+\infty} f(\tilde{x}) > 0$, and, hence, $\frac{\partial M}{\partial w_0} < 0 \Rightarrow \frac{da^*}{dw_0} < 0$, i.e.: For IARA utility, the amount invested in a risky project decreases in agent's initial wealth; risky projects are inferior goods.

5.1.4 Comparative Statics II: The Effect of a Change in i

$$\frac{\partial M}{\partial i} = E[-u'(y_f) + u''(y_f)(w_0 - a^*)(\tilde{x}-i)]$$

$$\Rightarrow \frac{da^*}{di} = -\frac{\frac{\partial M}{\partial i}}{\frac{\partial M}{\partial a}} = -\frac{E[-u'(y_f)]}{\frac{\partial M}{\partial a}} - (w_0 - a^*)\frac{E[u''(y_f)(\tilde{x}-i)]}{\frac{\partial M}{\partial a}}$$

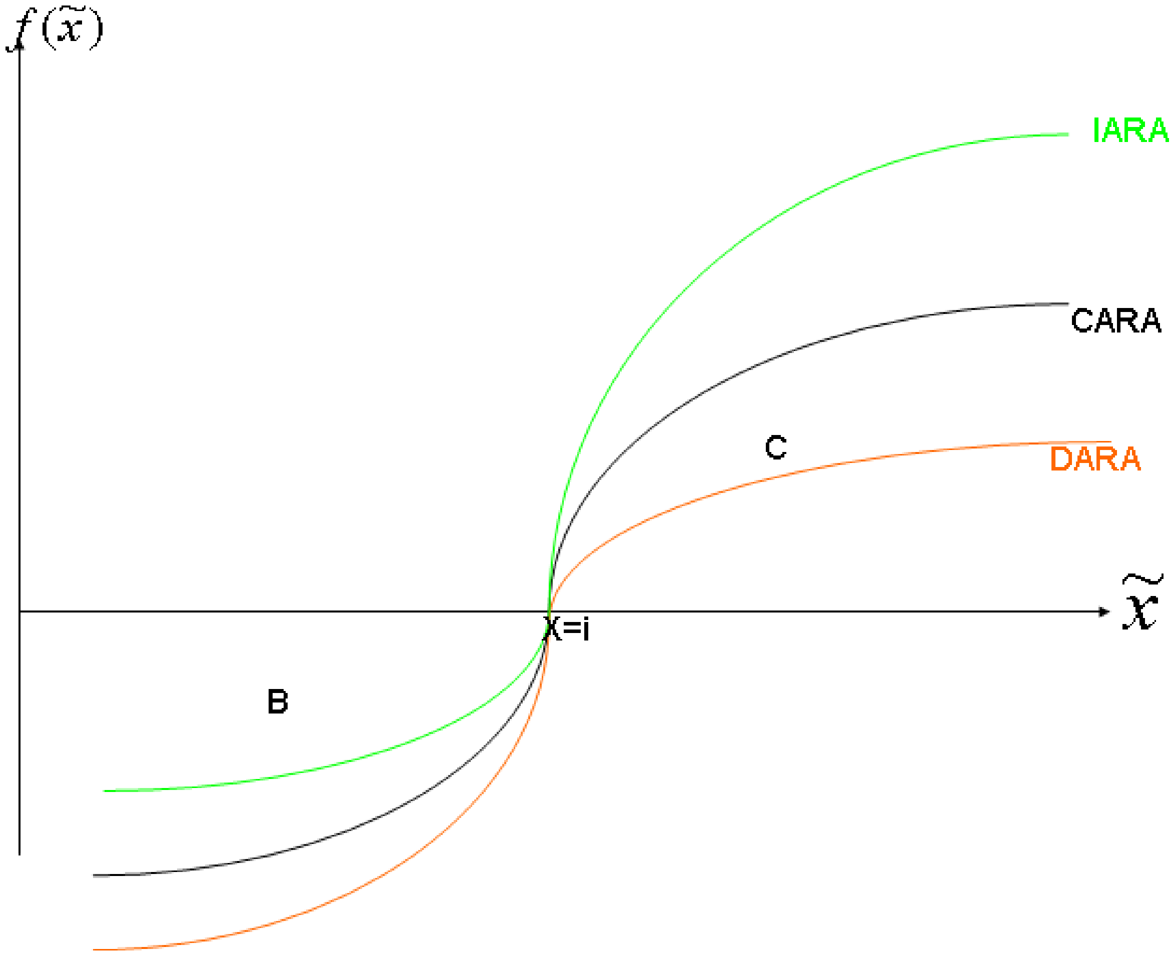


Figure 13: The Impact of Agent's Initial Endowment w_0 on the Demand for Risky Investments

$$\Longleftrightarrow \frac{da^*}{di} = \frac{E[u'(y_f)]}{\frac{\partial M}{\partial a}} + \frac{w_0 - a^*}{1+i}\frac{da^*}{dw_0}$$

This last equality amounts to a Slutsky—Equation, where the first term, always negative, gives the substitution effect, whereas the second term, the sign of which is ambiguous, measures the wealth/income effect. E.g.: If $\frac{da^*}{dw_0} > 0$ (DARA) and $(w_0 - a^*) < 0$, then $\frac{da^*}{di} < 0$. Intuition: $(w_0 - a^*) < 0$ means agent is selling the riskless investment; rising i implies agent is getting poorer; because of DARA, agent will become more risk–averse, and, hence, will be buying less of the risky investment.

5.1.5 Comparative Statics III: Increase in Risk Intensity

Let there be two risky assets, $\tilde{x}$ and $\tilde{y}$, distributed according to density functions $f(s)$ and $g(s)$, respectively, where s is a realization of $\tilde{x}$ or $\tilde{y}$, respectively. Suppose furthermore that $\int_A^B sf(s)\,ds = \int_A^B sg(s)\,ds$ (i.e. equality of mean), and that $\forall x \in [A; B] : \int_A^x [G(s) - F(s)]\,ds \geq 0$ (i.e. $F \succ^{SOSD} G$).

Now, define a^* so that the FOC hold for density f:

$$\int_A^B u'(w_0(1+i) + a^*(s-i))(s-i)f(s)\,ds = 0$$

Since distribution g is, as it were, "riskier" (on account of SOSD), we should expect that:

$$\int_A^B u'(w_0(1+i) + a^*(s-i))(s-i)g(s)\,ds < 0$$

i.e. that risk–averse agent will invest less if and when the project gets riskier. Combining the two equations from above, that is the case iff

$$\int_A^B u'(w_0(1+i) + a^*(s-i))(s-i)[g(s) - f(s)]\,ds < 0$$

Now, define $v(s) \equiv u'(w_0(1+i) + a^*(s-i))(s-i)$. Now, recall the Rothschild–Stiglitz theorem (1970) from Chapter 4: The equality above will hold iff v is a utility function for a risk–averse agent, i.e. iff v is concave in s. So let us check just that:

$$\frac{d}{ds}[u'(w_0(1+i) + a^*(s-i))(s-i)] = u''(.)(s-i)a^* + u'(.)$$

$$\frac{d^2}{ds^2}[u'(w_0(1+i) + a^*(s-i))(s-i)] = 2u''(.)a^* + u'''(.)(s-i)(a^*)^2$$

If we rule out strictly negative a^*, we know that $2u''(.)a^* \leq 0$. However, $sgn[u'''(.)(s-i)(a^*)^2] = ??$.

Hence, the change, as well its sign, in the amount risk–averse agent will invest in a risky project if risk becomes more intense (without there being a change in expected value), is *ambiguous*.

The appertainig literature has developed a plethora of reactions to this surprising result, such as e.g.

1. It is a good thing our theory should allow for such ambiguities (as with Giffen goods). Indeed, the more ambiguous a theory is, the more empirical data can be fitted into its

framework. Intuitively: If a project becomes more risky, risk–averse agent, shunning risk, will switch out of the risky asset (substitution effect). However, supposing the risky project pays out more in expectation than the riskless asset (which, as we recall, is both necessary and sufficient for any risk–averse agent at all to partake in the risky project), switching out of the risky project makes agent poorer. Thus, there could be an income effect, which might work in the opposite direction.

2. Impose restrictions on the utility function: If $a^* > 0$, and the Pratt–Arrow–measure of partial risk–aversion is lower than 1 and increasing in s, then risk–averse agent will invest less in a risky project as risk becomes more intense.

$$R_p = -w_1 \frac{u''(w_0 + w_1)}{u'(w_0 + w_1)} = -(s-i)a^* \frac{u''(y_f)}{u'(y_f)}$$

$$\frac{\partial R_p}{\partial s} = -a^* \frac{u''}{u'} - (s-i)(a^*)^2 \frac{u'u''' - (u'')^2}{(u')^2}$$

$$= -\frac{1}{u'}[a^* u''(1 + R_p) + (s-i)(a^*)^2 u'''] > 0$$

Recall that if $2a^*u'' + (s-i)u'''(a^*)^2 \overset{!}{<} 0$, there is no ambiguity. From the above, we know that $a^*u''(1 + R_p) + (s-i)(a^*)^2 u''' < 0$. Since $u'' < 0$ and $R_p < 1$, it follows that $2a^*u'' + (s-i)u'''(a^*)^2 < 0$ holds. Note that whereas the hypothesis that $\frac{\partial R_p}{\partial w_1} > 0$ seems quite justifiable, the supposition that $R_p < 1$ is empirically dubious, as empirically $R_p \in [2; 3]$.

3. Restrictions imposed on the definition of what constitutes an increase in risk

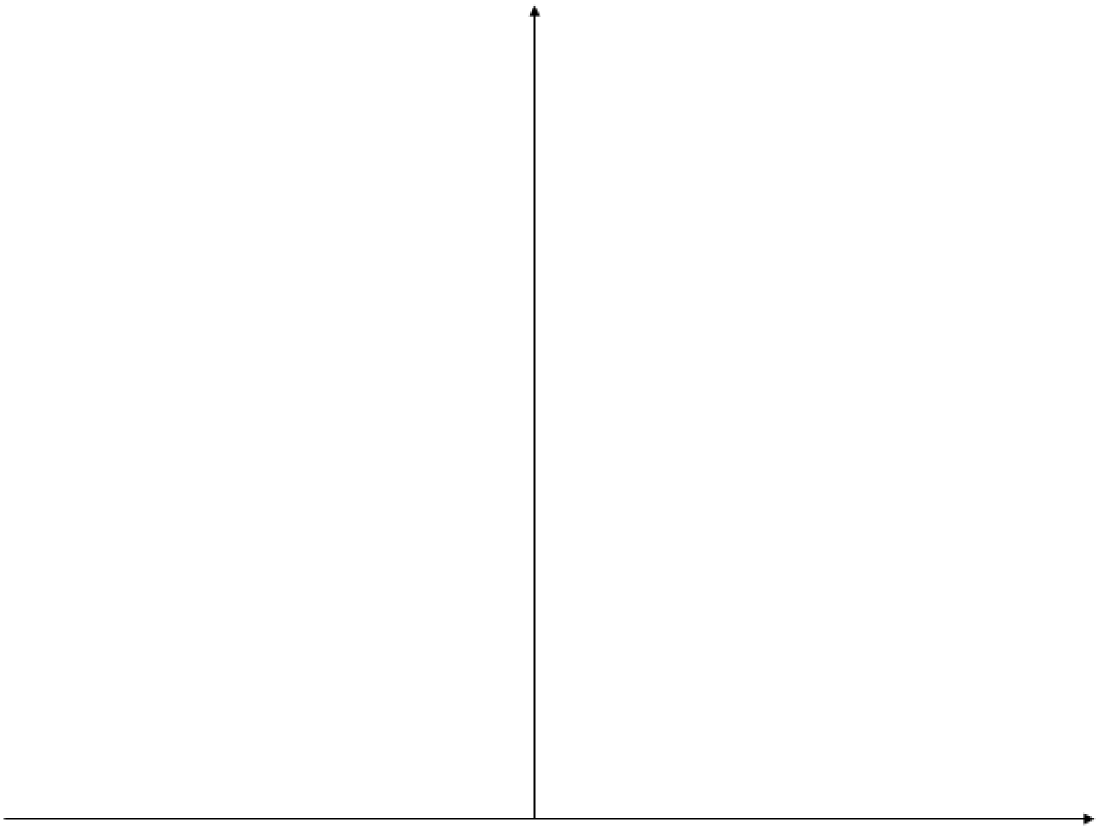
   - Strong increase in risk (Meyer & Ormiston, 1985)
   - μ—σ—approach: For DARA, CARA, and "small" IARA, agent will be investing less in a risky project if its risk, as measured by the variance of the distribution, increases (with expected value remaining equal); cf. Sinn, 1990.

4. Market equilibrium (Gollier & Schlesinger, 1997) One could expect the price of an asset to fall if the risk associated with it increases. However, this is only the case if another integral condition holds.

5.1.6 Comparative Statics IV: Increase in Risk Aversion

Recall the Pratt theorem from Chapter 3: u_{II} is more risk–averse than u_I iff $\exists G : u_{II} = G(u_I) \wedge G$ concave. The FOC for u_I is given by:

$$E[u_I'(w_0(1+i) + a^*(\tilde{x} - i))(\tilde{x} - i)] \overset{!}{=} 0$$



Now, for u_{II}:

$$\int_A^i G'(u_I(.))u_I'(w_0(1+i)+a^*(s-i))(s-i)f(s)\,ds + \int_i^B G'(u_I(.))u_I'(.)(s-i)f(s)\,ds$$

Note that the first term of the sum is negative, whereas the second term is positive. In addition, since $G'' < 0$:

$G'(u_I(w_0(1+i)+a^*(s-i)))$ (for $s > i$) $< G'(u_i(w_0(1+i))) < G'(u_I(w_0(1+i)+a^*(s-i)))$ (for $s < i$)

Hence, it follows that the sum of both terms is negative.

Thus: If agent gets more risk–averse she will invest less in risky projects.

5.2 The Demand for Insurance

Let w_0 be agent's initial wealth and u her utility function. With probability π, she suffers a loss of L. She can buy cover C at a premium rate of p. Thus, the premium she has to pay amounts to pC. Thus, with no loss occuring, her final wealth will be $w_1 \equiv w_0 - pC$, whilst, if a loss occurs, final wealth will amount to $w_2 \equiv w_0 - L + (1-p)C$.

$$\Rightarrow \max_C (1-\pi)u(w_0 - pC) + \pi u(w_0 - L + (1-p)C)$$

$$\begin{aligned} \text{FOC: } & -p(1-\pi)u'(w_0 - pC) + \pi(1-p)u'(w_0 - L + (1-p)C) \overset{!}{=} 0 \\ \iff & \frac{(1-\pi)u'(w_0 - pC)}{\pi u'(w_0 - L + (1-p)C)} \\ \iff & \frac{u'(w_1)}{u'(w_2)} = \frac{1-p}{1-\pi}\frac{\pi}{p} \end{aligned}$$

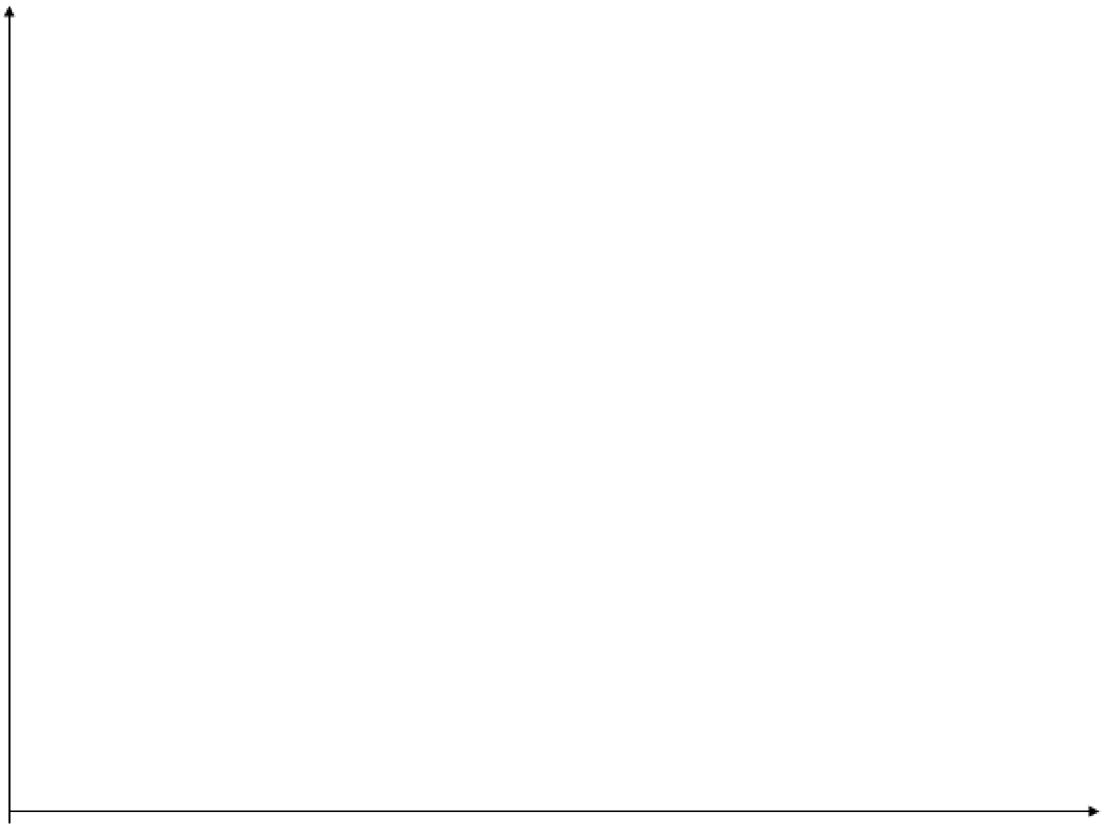


Figure 14: The Demand for Insurance

$$\frac{u'(w_1)}{u'(w_2)} = \frac{1-p}{1-\pi}\frac{\pi}{p} \begin{cases} < 1 & \text{if } p > \pi \Rightarrow w_1 > w_2 \iff C < L; \\ = 1 & \text{if } p = \pi \Rightarrow w_1 = w_2 \iff C = L; \\ > 1 & \text{if } p < \pi \Rightarrow w_1 < w_2 \iff C > L. \end{cases}$$

The insurance premium p is said to be "fair" if the insurance company makes 0 profits in expectation, i.e. if $E[G] = pC - \pi C = C(p - \pi) \stackrel{!}{=} 0 \iff p = \pi$.

Thus, any risk—averse agent who is offered insurance at a fair premium will buy full insurance. Note that this outcome is efficient since we have assumed the company to be risk—neutral and agent to be risk—averse.

5.3 Firms under Uncertainty

5.3.1 Introduction

For reasons of risk–spreading and risk–sharing, firms will usually be modeled as being risk–neutral agents. However, reasons why firms may be considered risk–averse are no less abundant:

- Agency Problems Within the Firm: As we know from the micro course, incentive pay will sometimes have to be used to overcome Moral Hazard in the owner—manager—relationship, i.e. the manager's pay will be made dependent on the firm's performance. This may in turn lead to a situation where, instead of maximizing the *risk–neutral owners'* expected return, *risk–averse* managers will maximize their own utility. Thus, as a result, the firm will behave like a risk–averse entity.
- Bankruptcy Costs: The threat of bankruptcy may lead to non–linearities in the pay–off schedule. This may be due to the loss of firm–specific human capital or to the loss of the customer base.
- Convex Tax Schedules: If the marginal tax rate is increasing, higher profit realizations become less valuable.

The uncertainty firms face may appertain either to production decisions or to investment choices.

5.3.2 Production Decisions

The uncertainty in production decisions may pertain either to market conditions (input factor prices or selling prices) or to technology (i.e. there is a stochastic element to the physical amount of output that is generated).

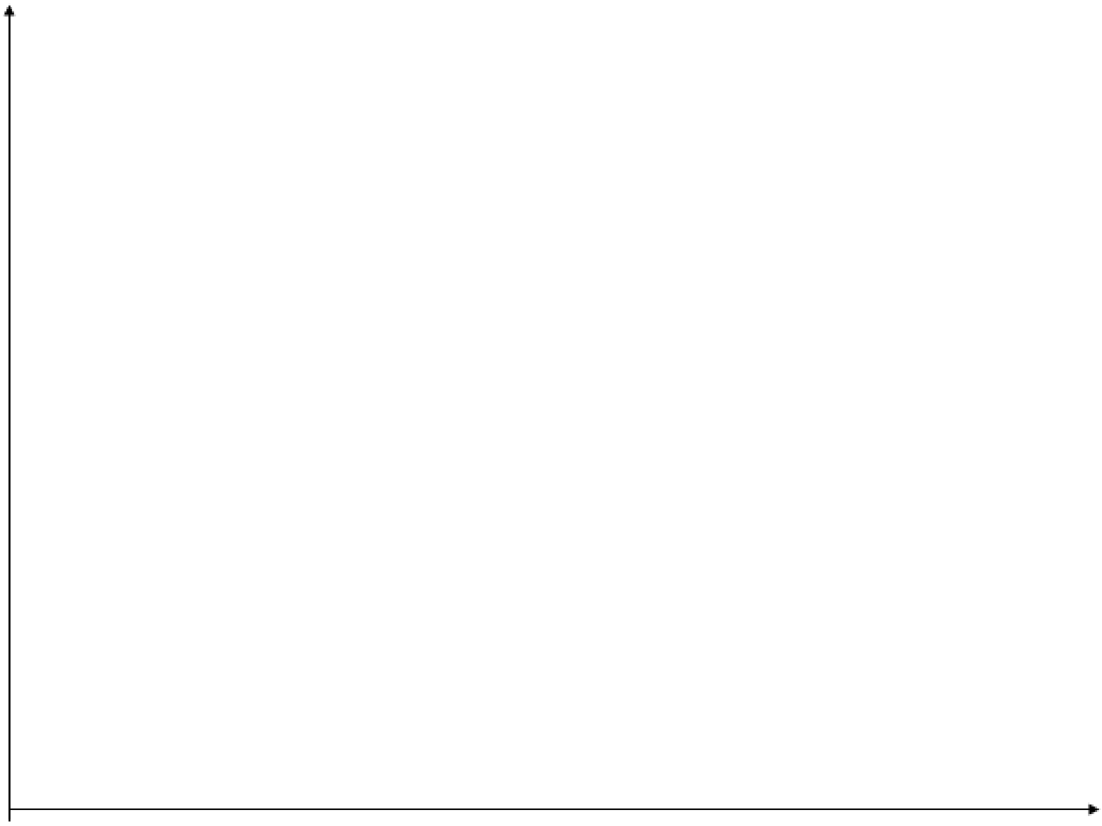


Figure 15: Bankruptcy Costs

a) Price Uncertainty

Let w_0 be the firm's initial wealth, a the amount of output produced, $c(a)$ the cost of producing output a, and $\tilde{p}$ the (uncertain) selling price. Then, owner's final wealth $\tilde{w_f}$ is given by $\tilde{w_f} = w_0 + \tilde{p}a - c(a)$. The risk–averse decision–maker (owner or manager) will maximize $u(\tilde{w_f})$.

$$\text{FOC: } \frac{dE[u(\tilde{w_f})]}{da} = E[u'(w_0 + \tilde{p}a - c(a))(\tilde{p} - c'(a))] \stackrel{!}{=} 0$$

Recall that

$$Cov(\tilde{x}, \tilde{y}) = E[\tilde{x}\tilde{y}] - E[\tilde{x}]E[\tilde{y}] \iff E[\tilde{x}\tilde{y}] = Cov(\tilde{x}, \tilde{y}) + E[\tilde{x}]E[\tilde{y}]$$

Hence, the FOC is equivalent to:

$$Cov(u'(.), (\tilde{p} - c'(a))) + E[u'(.)]E[\tilde{p} - c'(a)] \stackrel{!}{=} 0$$

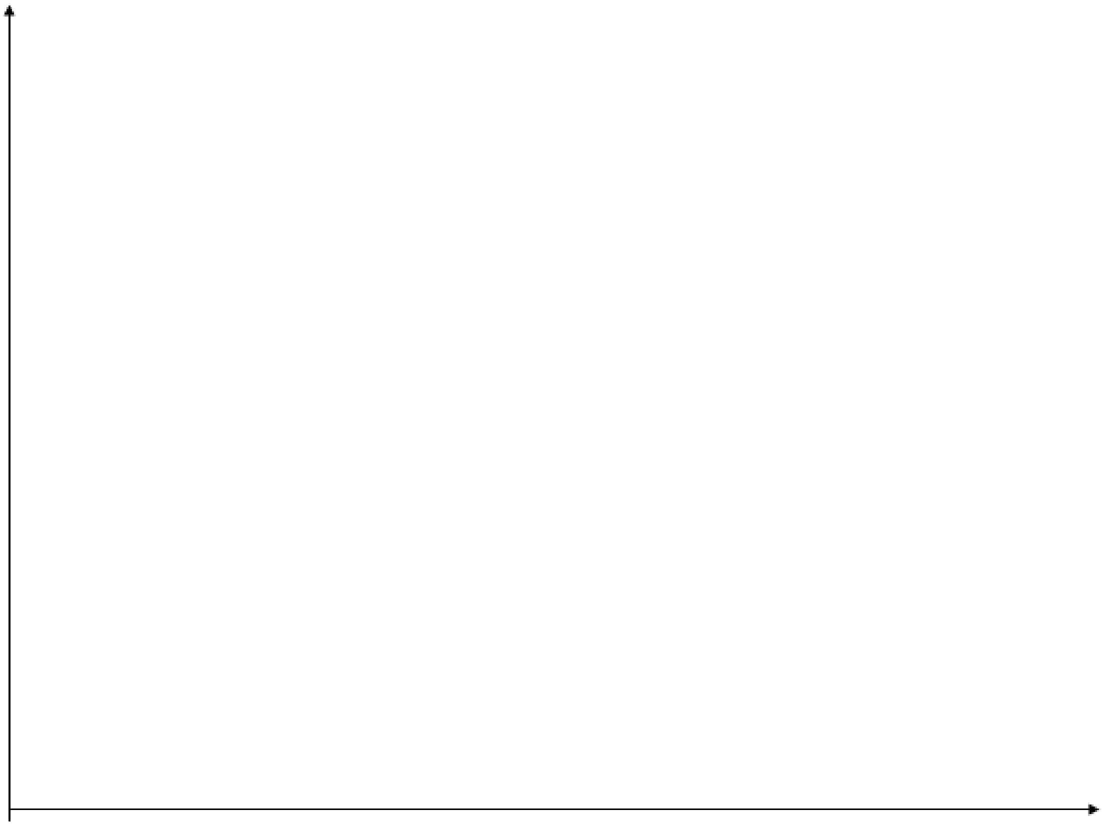


Figure 16: Convex Tax Schedule

Note that $c'(a)$ is not a random variable. Therefore, the FOC reduces to

$$Cov(u'(.), \tilde{p}) + E[u'(.)](E[\tilde{p}] - c'(a)) \stackrel{!}{=} 0$$

$$\iff E[\tilde{p}] = c'(a) - \frac{Cov(u'(.), \tilde{p})}{E[u'(.)]}$$

If the firm were risk–neutral this optimality condition would reduce to $E[\tilde{p}] = c'(a)$, since risk–neutral (linear) u implies $u' = const \Rightarrow Cov(u'(.), \tilde{p}) = 0$.

Note that if $\tilde{p}$ is high, final wealth $\tilde{w_f}$ is high, too, implying low $u'(\tilde{w_f})$. Thus, $sgn\{Cov(u'(.), \tilde{p})\} = -1$, implying $-\frac{Cov(u'(.),\tilde{p})}{E[u'(.)]} > 0$. One may think of $-\frac{Cov(u'(.),\tilde{p})}{E[u'(.)]}$ as some kind of additional "psychological" marginal cost.

In conclusion, *risk–aversion will reduce agent's willingness to produce given expected prices.*

Comparative static analysis yields results that are perfectly analogous to those we had been discussing under the header of optimum portfolio selection. Be aware that in the case of fixed costs, i.e. of $c(a) = c_0 + c_1(a)$, an increase in fixed costs is tantamount to a decrease in agent's initial wealth. Thus, e.g. producers exhibiting DARA—utility will reduce production if fixed costs increase. It can also be shown that psychological marginal costs will be increasing if fixed costs are increasing, i.e. $\frac{d}{dc_0}[-\frac{Cov(u'(.),\tilde{p})}{E[u'(.)]}] \geq 0$.

b) Technological Uncertainty

One can distinguish two forms of technological uncertainty: additive and multiplicative.

α) Additive Uncertainty

$$\tilde{w_f} = w_0 + p(a + \tilde{\epsilon}) - c(a)$$

$$\max_a E[u(\tilde{w_f})]$$

$$\Rightarrow \text{ FOC: } \frac{d}{da}[E[u(\tilde{w_f})] = E[u'(\tilde{w_f})(p - c'(a)) \stackrel{!}{=} 0$$

$$\iff p = c'(a) \qquad\qquad (\text{indeed } u'(\tilde{w_f}) > 0 \,\forall \tilde{w_f})$$

The choice of a does not have any impact on the uncertainty $p\tilde{\epsilon}$. Hence, a will be chosen as though there were no uncertainty.

β) Multiplicative Uncertainty

$$\tilde{w_f} = w_0 + p[a(1 + \tilde{\epsilon})] - c(a)$$

$$= w_0 + p(1 + \tilde{\epsilon})a - c(a)$$

$$= w_0 + \tilde{p}a - c(a) \qquad\qquad \text{with } \tilde{p} \equiv p(1 + \tilde{\epsilon})$$

Thus, from a formal point of view, this problem is perfectly analogous to the problem of price uncertainty: same problem, same results!

Note that, in contrast to additive uncertainty, here the choice of a has an impact on the intensity of the uncertainty faced. Indeed, if initially chosen a is low, the effect of the shock $\tilde{\epsilon}$ is weak. Conversely, if initially chosen a is high, the effect of the shock $\tilde{\epsilon}$ is strong.

5.3.3 Investment Decisions

Let there be one (irreversible) investment project, which is engendering one-time costs of I. Its payoff $\tilde{\pi}(t)$, $t = 1, \cdots$ we'll suppose to be uncertain.

Orthodox theory would propose the following decision rule: Invest iff NPV[3] ≥ 0,

$$\iff E[\sum_{t=1}^{T} \delta^t \tilde{\pi}(t)] - I \geq 0$$

where $\delta \equiv \frac{1}{1+r}$ is the discount factor and r the (real) interest rate.[4]

However, it can be argued that our theory should be mindful of whether an investment project could be delayed, canceled, or interrupted. Indeed, this consideration will lead us to the Real Option Theory of Investment. It is so named because it considers the possibility of agent's doing something "real", such as waiting, for instance.

Example

- $I = 1600$ \$
- $r = 10\%$
- In $t = 0$, the project will certainly yield a payoff of 200 \$
- In $t \in [1; +\infty]$, the project will with equal probability yield a payoff of either 100 \$ or 300 \$

$$NPV = E[\sum_{t=0}^{\infty} \delta^t \tilde{\pi}(t)] - I$$

$$= 200 + \sum_{t=1}^{\infty} \delta^t \frac{300 + 100}{2} - I = \sum_{t=0}^{\infty} \frac{200}{(1,1)^t} - 1600$$

$$= \frac{200}{1 - \frac{1}{1,1}} - 1600 = 600 > 0$$

Thus, using the NPV—criterion, the project ought to be engaged in. But, what if agent waited till tomorrow ($t = 1$)?

If $\pi(t = 1) = 100 \Rightarrow NPV(t = 1) = 1100 - 1600 < 0$. Thus, in this case, the project should be refrained from.

If $\pi(t = 1) = 300 \Rightarrow NPV(t = 1) = 3300 - 1600 > 0$. Thus, in this latter case, the project should be commenced.

[3]Net Present Value

[4]As we know from Macro, investment decisions are dependent on the *real* interest rate, wheras portfolio decisions depend on nominal interest

Now, how would the following project be valued: Wait for 1 period; invest iff $\pi(t = 1) = 300$?

$$NPV(t = 0) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \frac{NPV(t = 1)}{1,1} = \frac{1}{2} \cdot \frac{1700}{1,1} = 773$$

Remember that the project where agent had to make a decision in $t = 0$ had a $NPV(t = 0) = 600$. Hence, the flexibility option F is valued at $F = 173\$$. With flexibility, there are indeed three effects:

- Agent will lose 200\$ in $t = 0$
- Agent will gain $I(1 - \delta)$, since investment costs become due one period later. Hence, $I \uparrow \Rightarrow F \uparrow$
- Agent will get 300\$ instead of $\frac{300+100}{2}$ (in expectation), since she need no longer bear the uncertainty

Note that in our example $p_0 = 200$ \$ and $p_1 = 1,5p_0$ or $p_1 = 0,5p_0$ with equal probability. For $p_0 < 97$ it is optimal never to invest, whereas for $p_0 > 249$ it is optimal to invest as soon as $t = 0$. For $97 < p_0 < 249$, it is optimal to wait and invest only if $p_1 = 1,5p_0$.

With a mean—preserving spread (such as e.g. $\{100; 300\} \mapsto \{50; 350\}$), $F \uparrow$, since $p_1 = 100$ and $p_1 = 50$ are both equivalent as both lead to $NPV(t = 1) = 0$, as no investment will take place.

Note that in this context waiting is perfectly equivalent to buying a signal. In our setup, agent buys the signal by foregoing a certain profit of 200\$ in $t = 0$. As we shall see in greater detail in a later chapter, a rational decision maker should never seek costly information unless there is a chance that information may actually change what she is going to do.

Real Option Theory can be used to analyze a plethora of situations:

- Interest Rates: The theory would suggest that uncertainty had more of an impact on investment decisions than the current interest rate.
- Labor Markets: Hiring & firing costs are analogous to investment or cancelation costs. Our theory would thus predict that a high measure of uncertainty led to a decline in new job offers.
- Hysteresis effects, e.g. uncertainty relating to currency exchange rates For instance, from 1980–1984, the U.S. dollar rose, as did U.S. imports. From 1984–1987, the dollar again fell to 1980 levels, yet imports hardly budged.

- Oil Reserves
- Product Development (e.g. electric cars)
- R & D
- Marriage
- Suicide
- Changes in the Law

Economics of Information and Uncertainty

Summer Term 2006

6 Allocation of Risk

6.1 Efficient Risk Allocation

Let there be a simple exchange economy with two individuals (1 and 2) and two possible states of the world (a and b) that realize with probabilities p and $1-p$, respectively. Let furthermore individual j's initial endowment in state i be denoted by $w_{0j}(i)$, and her final wealth (i.e.after trade) in state i by $w_{fj}(i)$. Individual j's utility is assumed to be given by the at least twice differentiable function u_j, with $u_j' > 0 > u_j''$.

Social planner's problem:

$$\max_{w_{f1}(a),w_{f1}(b),w_{f2}(a),w_{f2}(b)} pu_1(w_{f1}(a)) + (1-p)u_1(w_{f1}(b)) \text{ s.t.}$$

$$pu_2(w_{f2}(a)) + (1-p)u_2(w_{f2}(b)) \geq pu_2(w_{02}(a)) + (1-p)u_2(w_{02}(b)) \equiv \bar{u}$$

$$w_{f1}(a) + w_{f2}(a) \leq w_{01}(a) + w_{02}(a) \equiv w_0(a)$$

$$w_{f1}(b) + w_{f2}(b) \leq w_0(b)$$

$$\iff \max_{w_{f2}(a),w_{f2}(b)} pu_1(w_0(a) - w_{f2}(a)) + (1-p)u_1(w_0(b) - w_{f2}(b)) \text{ s.t.}$$

$$pu_2(w_{f2}(a)) + (1-p)u_2(w_{f2}(b)) \geq \bar{u}$$

Langrangian maximization yields:

$$\text{FOC:} \frac{pu_1'(a)}{(1-p)u_1'(b)} = \frac{pu_2'(a)}{(1-p)u_2'(b)}$$

(The SOC holds because of risk aversion ($u'' < 0$).)

This result holds in the general case and is commonly referred to as the *Borch Condition: An allocation of risk is Pareto–efficient iff, in all possible states of the world, the marginal rate of substitution of income in state s and income in state t is equalized over all individuals:*

$$\forall i, j, s, t : \frac{p_s u_i'(w_{fi}(s))}{p_t u_i(w_{fi}(t))} = \frac{p_s u_j'(w_{fj}(s))}{p_t u_j'(w_{fj}(t))}$$

With risk being allocated optimally, both parties will bear some risk if

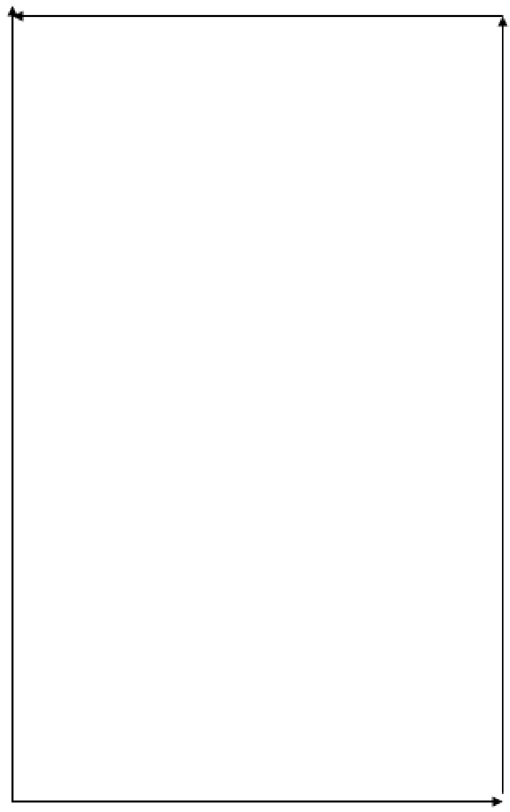


Figure 17: A simple echange economy with social risk

- there is social risk AND
- both parties are risk–averse

Suppose individual 2 is risk–neutral (implying that $u_2' = const$). Then, the Borch Condition simplifies to:

$$\frac{pu_1'(a)}{(1-p)u_1'(b)} = \frac{pu_2'(a)}{(1-p)u_2'(b)} = \frac{p}{1-p}$$

Since 1 is still risk–averse, $u_1'(a) = u_1'(b) \Rightarrow w_{f1}(a) = w_{f1}(b)$. Hence, the risk–neutral agent will fully insure the risk–averse agent, which intuitively makes great sense since the social planner's goal is to minimize the risk costs agents suffer and risk–neutral agents do not bear any costs from having to shoulder risk.

Theorem 6.1 The Reciprocity Principle: *In any Pareto–efficient risk allocation, an individual's final wealth is dependent ONLY on society's total wealth in the respective state of*

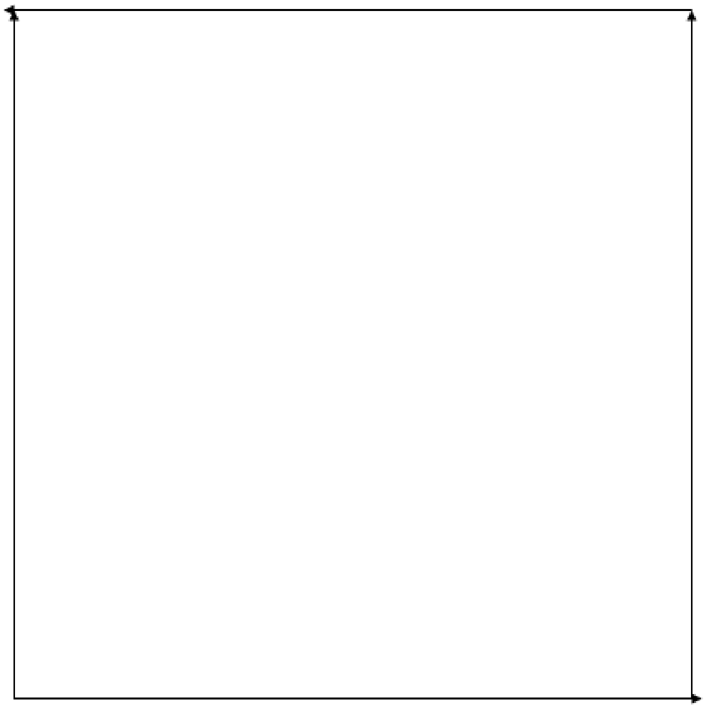


Figure 18: A simple exchange economy with no social risk

the world.

$$w_{fj}(i) = F_j(w_0(i))$$

Proof It has to be proved that $w_0(s) = w_0(t) \Rightarrow w_{fj}(s) = w_{fj}(t) \forall j$.

Suppose $w_0(s) = w_0(t)$, but $\exists j$ s.t. $w_{fj}(s) > w_{fj}(t) \iff u'[w_{fj}(s)] < u'[w_{fj}(t)]$ (*), due to risk–aversion ($u'' < 0$).

However, in a Pareto–efficient equilibrium, the Borch Condition will apply:

$$\frac{p_s u'_j[w_{fj}(s)]}{p_t u'_j[w_{fj}(t)]} = \frac{p_s u'_k[w_{fk}(s)]}{p_t u'_k[w_{fk}(t)]}$$

From (*), it follows that $LHS < 1 \Rightarrow RHS < 1 \Rightarrow u'_k[w_{fk}(s)] < u'_k[w_{fk}(t)] \iff w_{fk}(s) > w_{fk}(t)$ due to risk–aversion.

Thus, in a Pareto–efficient equilibrium, it must be the case that $w_{fj}(s) > w_{fj}(t)$ implies

$w_{fk}(s) > w_{fk}(t)$, and, summing over all the individuals

$$w_0(s) = \sum_q w_{fq}(s) > \sum_q w_{fq}(t) = w_0(t)$$

which contradicts our assumption that $w_0(s) = w_0(t)$ and thus completes the proof. ∎

The Reciprocity Principle immediately implies that if there is no social risk, all individuals are fully insured.

Now, let us examine the question of who will optimally bear what proportion of the social risk, i.e. of how $F_j(w_0(i))$ will change in $w_0(i)$. Suppose there are two individuals with $u_1'(F_1(w_0)) = \lambda u_2'(F_2(w_0))$ (*). Total differentiation and division by dw_0 yields:

$$u_1''(F_1(w_0))F_1'(w_0) = \lambda u_2''(F_2(w_0))F_2'(w_0) \ (^{**})$$

$$(^{**})/(^{*}) \Rightarrow F_1'(w_0)\frac{u_1''(.)}{u_1'(.)} = F_2'(w_0)\frac{u_2''(.)}{u_2'(.)}$$

Remember the Pratt–Arrow–coefficient of absolute risk aversion $A(y) \equiv -\frac{u''(y)}{u'(y)}$

$$\iff A_1(F_1(w_0))F_1'(w_0) = A_2(F_2(w_0))F_2'(w_0)$$

Since there are only two individuals in this economy, $F_1'(w_0) + F_2'(w_0) = \frac{dw_0}{dw_0} = 1$

$$\Rightarrow A_1F_1' = A_2(1 - F_1')$$

$$\Rightarrow F_1'(w_0) = \frac{A_2}{A_1 + A_2} = \frac{A_1^{-1}}{A_1^{-1} + A_2^{-1}} = \frac{T_1(w_{f1})}{T_1(w_{f1}) + T_2(w_{f2})}$$

with $T_j \equiv A_j^{-1}$ denoting the individual j's coefficient of risk tolerance. In general:

$$\frac{dw_{fj}}{dw_0} = \frac{T_j(w_{fj})}{\sum_{q=1}^{n} T_q(w_{fq})}$$

Thus, in Pareto-optimum, the larger an individual's share of society's total risk tolerance the stronger the impact of a change in society's total wealth on that individual's final endowment.

Note that if there is no social risk, w_0 is constant by definition.

Suppose individual 2 were risk–neutral. Then $A_2 = 0 \Rightarrow T_2 = \infty \Rightarrow \frac{dw_{f1}}{dw_0} = \frac{T_1}{T_1+T_2} = 0$, meaning that individual 1 is fully insured. If in an economy, one individual is risk–neutral, that individual optimally takes on all of the social risk, thus fully insuring all the other individuals in the society.

If all individuals in an economy exhibit CARA—utility, $A_j = const_j \forall j$, it would follow that $\frac{dw_{fj}}{dw_0} = const$, implying a linear sharing rule: $w_{fj}(w_0) = a_j + b_j w_0$.

It is routine to verify that quadratic utility of the form $u_j(w_{fj}) = w_{fj} - \beta_j w_{fj}^2$ also implies a linear sharing rule.

Generically, though, optimal sharing rules will be non—linear.

6.2 How and When can Efficiency in Risk Allocation Be Achieved?

6.2.1 Arrow Securities

An Arrow Security a_s for state of the world s is defined as a security that pays out unity in state of the world s and 0 in all other states of the world. Let q_s denote the market price of one Arrow Security for state s. If we assume competitive markets and there is no discounting of future payments, arbitrage will lead to $\sum_s q_s = 1$. Let $x_{fj}(s)$ denote individual j's final wealth in state of the world s, which results from her initial endowment as well as from trading with Arrow Securities.

Individuals will maximize

$$\sum_s p_s u(x_{fj}(s)) \text{ s.t.}$$

$$\sum_s q_s(x_{fj}(s) - x_{0j}(s)) \leq 0$$

$$\text{FOC for state s: } p_s u'(x_{fj}(s)) - \lambda_j q_s \stackrel{!}{=} 0$$

$$\text{FOC for state t: } p_t u'(x_{fj}(t)) - \lambda_j q_t \stackrel{!}{=} 0$$

Dividing both FOCs yields:

$$\frac{p_s u'(x_{fj}(s))}{p_t u'(x_{fj}(t))} = \frac{q_s}{q_t}$$

Thus, in words, the market will lead to a result where MRS=price ratio. By our assumption of competitive markets, the Law of One Price is implied, meaning that this price ratio will be the same for all market participants. Thus, in equilibrium, marginal rates of substitution will equalize and the Borch Condition will hold. Thus, with complete markets for Arrow Securities, *the market will replicate the benevolent social planner's solution.*

The price vector q is given by the market clearing conditions (MCC):

$$\forall s : \sum_j (x_{fj}(s) - x_{0j}(s)) \leq 0$$

If there is but a single risk—neutral individual in the market, prices will be fair, meaning $\forall s : q_s = p_s$.

If there are several goods, we can analyze "contingent claims". Let q_{es} denote the price of good e in state s.

$$\max \sum_s p_s u(x^1_{fj}(s), x^2_{fj}(s), \cdots) \text{ s.t.}$$

$$\sum_s \sum_e q_{es}(x^e_{fj}(s) - x^e_{0j}(s)) \leq 0$$

$$\Rightarrow p_s u_1(...) - \lambda_j q_{1s} \overset{!}{=} 0 \text{ etc.}$$

The same result can be achieved if there are only Arrow Securities and goods are exchanged only after the relevant state of the world has realized, provided individuals know the prices of the goods when the Arrow Securities are traded.

Production Decisions

Let $f_{ej}(a, s)$ denote firm j's output of good e in state of the world s resulting from investment a.

$$\max_a \pi_j(a) = \sum_s \sum_e q_{es} f_{ej}(a, s) - a$$

Note that there is no uncertainty about the firm's profit since state—contingent goods are traded *today*, i.e. before the state of the world realizes. Since there is no uncertainty, all the shareholders would choose the same a; investment decisions are made by unanimous consent of all the firm's shareholders. All trades are done today; under the assumptions from above, we thus get rid of all idiosyncratic risks.

Conclusion:

Complete markets with Arrow Securities lead to a Pareto—efficient allocation of risk; i.e. the 1st Main Theorem of Welfare Economics holds.

6.2.2 Real—World—Securities

Real—world markets for Arrow Securities can be found in many places, as e.g. insurance markets, equity markets...

Consider a security (e.g. a stock) j of price P_j, and let π_{js} denote the stock j's payoff in state s. We know that, because of arbitrage, the stock should cost just as much as it would cost to replicate the stock's payoff vector with Arrow Securities, thus

$$P_j = \sum_s q_s \pi_{js}$$

Thus, if P is the matrix of all available securities $j \in \{1, \cdots, J\}$ and q the price vector of (hypothetical) Arrow Securities for all possible states of the world $s \in \{1, \cdots, S\}$, and π the $J \times S$—dimensional matrix of all π_{js}

$$P = \pi q \Rightarrow q = \pi^{-1} P$$

This tells us that a full market with Arrow Securities can be replicated iff matrix π is invertible, which is the case if the number of linearly independent assets is equal to the number of possible states of the world. Hence, markets will lead to an efficient risk allocation if there are enough linearly independent assets.

6.2.3 Incomplete Markets

Suppose there are two states of the world s_1 and s_2 but only one asset b_1 that pays out 3 if $s = s_1$ and 1 if $s = s_2$. Let λ_j denote the share individual j holds in the firm b_1.

$$E[u_j] = pu(w_{fj}(s_1)) + (1-p)u(w_{fj}(s_2)) \text{ s.t.}$$

$$w_{fj}(s_1) = w_{0j}(s_1) + \lambda_j \cdot 3$$
$$w_{fj}(s_2) = w_{0j}(s_2) + \lambda_j \cdot 1$$

It is routine to verify that generically MRS will not equalize across individuals. Thus, efficient risk allocation will generically not be achieved. But is there a way to make the market complete? Consider a call option o_1 with a strike price of 2. Its payoff will thus be 1 in s_1 and 0 in s_2. Thus o_1 and b_1 are linearly independent and therefore they constitute a complete market, there being only two possible states of the world in this economy.

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7 The Demand for Information

7.1 Some Notation

Let $z_1, z_2, z_3, \cdots$ denote the states of the world that realize with an *ex ante*—probability of $w_1, w_2, w_3, \cdots$, respectively, with $\sum_i w_i = 1$. Let there be a system of signals $s_1, s_2, s_3, \cdots$, with π_k denoting the probability signal s_k will realize. Furthermore, let w_{ik} denote the *ex post*—probability of state z_i conditional on signal s_k having realized. Again, $\sum_i w_{ik} = 1$.

- Now let $p_{ik} \equiv Pr[z_i \wedge s_k]$ denote the *common probability* of state z_i and signal s_k. Then, $\sum_i p_{ik} = \pi_k$ and $\sum_k p_{ik} = w_i$.
- Let $q_{ik} \equiv Pr[s_k \mid z_i] = \frac{p_{ik}}{w_i}$ denote the *conditional probability* of signal s_k given state z_i.
- We have already defined $w_{ik} \equiv Pr[z_i \mid s_k] = \frac{p_{ik}}{\pi_k}$, the *ex post*—probability of state z_i given signal s_k.

7.2 Recap: Bayes's Rule

Remember the definition of conditional probabilities:

$$Pr[A \mid B] = \frac{Pr[A \wedge B]}{Pr[B]}$$

where $(A; B) \subset \Omega^2$ is a pair of events and Ω is the set of all possible events. From the above equation it follows immediately that

$$Pr[B \mid A] = \frac{Pr[B \wedge A]}{Pr[A]}$$

Define $\overline{A} \equiv \Omega \setminus A$, and, analogously $\overline{B} \equiv \Omega \setminus B$.

$$\begin{aligned} Pr[A] &= Pr[A \wedge B] + Pr[A \wedge \overline{B}] \\ &= Pr[A \mid B]Pr[B] + Pr[A \mid \overline{B}]Pr[\overline{B}] \end{aligned}$$

From these considerations, *Bayes's Rule* immediately follows:

$$Pr[B \mid A] = \frac{Pr[A \mid B]Pr[B]}{Pr[A \mid B]Pr[B] + Pr[A \mid \overline{B}]Pr[\overline{B}]}$$

7.3 The Value of a Signal System

7.3.1 Action is Chosen *Before* Receipt of Signal

Suppose agent must choose an action before receiving a signal. Let her choose a^*, leading to payoffs $x_1, x_2, x_3, \cdots$ in the states $z_1, z_2, z_3, \cdots$. Without a signal, expected utility is given by

$$E[u] = \sum_i w_i u(x_i)$$

With the benefit of the signal, expected utility is

$$\begin{aligned} E[u] &= \sum_k \pi_k \cdot \sum_i w_{ik} u(x_i) \\ &= \sum_k \sum_i \pi_k w_{ik} u(x_i) \\ &= \sum_k \sum_i \pi_k \frac{p_{ik}}{\pi_k} u(x_i) \\ &= \sum_i w_i u(x_i) (\textstyle\sum_k p_{ik} = w_i) \end{aligned}$$

which, as we have computed above, is equal to the expected utility without a signal.

Thus, as should hardly be surprising, *a signal received after the choice of action is worthless.*

7.3.2 Action is Chosen *After* Receipt of Signal

Let x_{li} denote the payoff resulting from action a_l if state z_i realizes.

Agent's problem without her having the benefit of a signal:

$$\max_{a_l} \sum_i w_i u(x_{li})$$

Without loss of generality, let us assume action $a^* = a_1$ maximized agent's expected utility.

Agent's problem after having received a signal s_k:

$$\max_{a_l} \sum_i w_{ik} u(x_{li})$$

Quite possibly, $a_k^* \neq a^*$, i.e. agent's optimal action conditional on her having received signal s_k may differ from her optimal action if there is no signal.

Note that, from Bayes's Rule, $w_i = \sum_k p_{ik} = \sum_k \pi_k w_{ik}$.

Let V denote the value of the signal system in utility terms:

$$V = [\sum_k \pi_k [\sum_i w_{ik} u(x_{k^*i}) - \sum_i w_{ik} u(x_{1i})]] \geq 0$$

If, for all s_k, $a_k^* = a^*$, then $V = 0$; otherwise $V > 0$. As would seem obvious, a signal's value is strictly positive iff it leads to a change of action.

Now, let G^* denote the monetary value of the signal system. G^* is implicitly given by

$$\sum_k \sum_i w_{ik} u(x_{k^*i} - G^*) = \sum_i w_i u(x_{1i})$$

i.e. agent is indifferent between not having the signal system and having the signal system but having to pay G^*.

Example Let there be a risk–neutral oil company interested in acquiring land for drilling. With equal probability there is oil or there is none. If the company does not drill, its certain payoff will be 0. If it chooses to drill and strikes oil, its payoff is 3. If it drills and does not strike oil its payoff is -1,5.

Expected profit if there is no signal: $\frac{1}{2} \cdot (-1,5) + \frac{1}{2} \cdot 3 > 0$, so the company should get drilling.

Now, suppose the company could test–drill for oil.

	z_1 (no oil)	z_2 (oil)	..
s_1 (signal bad)	0,3	0,1	$\pi_1 = 0,4$
s_2 (signal good)	0,2	0,4	$\pi_2 = 0,6$
	$\frac{1}{2}$	$\frac{1}{2}$	

From this we can easily compute the *ex post*—probabilities by using Bayes's Rule:

	z_1 (no oil)	z_2 (oil)	..
s_1 (signal bad)	0,75	0,25	
s_2 (signal good)	0,33	0,67	

If the signal it receives is s_1, the expected revenue from drilling is equal to $\frac{3}{4} \cdot (-1,5) + \frac{1}{4} \cdot 3 < 0$. So it is optimal not to drill then.

If, by contrast, it were to receive the signal s_2, expected revenue from drilling would amount to $\frac{1}{3} \cdot (-1,5) + \frac{2}{3} \cdot 3 > 0$. Thus, drilling would now be optimal.

Hence V is given by

$$V = 0,4(0 - [\frac{3}{4} \cdot (-1,5) + \frac{1}{4} \cdot 3]) + 0,6 \cdot 0 = 0,15$$

7.4 Blackwell Garbling

Let there be two signal systems S^1 and S^2, with S^r being defined by the realization of signals $s_1^r, s_2^r, \cdots$, occurring with probability $\pi_1^r, \pi_2^r, \cdots$ respectively. Signal system S^r thus leads to *ex post*—probabilities w_{ik}^r.

Definition A signal system (an informational structure) S_1 is said to be more valuable than informational structure S_2 iff the following condition holds for all individuals:

$$\sum_k \pi_k^1 \sum_i w_{ik}^1 u(x_{k_1^* i}) \geq \sum_k \pi_k^2 \sum_i w_{ik}^2 u(x_{k_2^* i})$$

Theorem 7.1 *S_1 is more valuable than S_2 iff there exist nonnegative numbers $\beta_{k'k}$ such that the following two conditions hold*

$$\forall k' : q_{ik'}^2 = \sum_k \beta_{k'k} q_{ik}^1$$

$$\forall k : \sum_{k'} \beta_{k'k} = 1$$

where $q_{ik} = Pr[s_k \mid z_i]$.

This theorem harks back to Blackwell (1951). Its proof being rather involved, we shall skip it here. Intuition, by contrast, is quite simple: What the theorem says is basically that $S_2 = S_1 +$ white noise, meaning any time there is a signal s_k^1, that signal is blurred or "garbled" by some stochastic process that is independent of the true state of the world. I.e. $q_{ik'}^2 = \sum_k Pr[k' \mid k] q_{ik}^1$.

Example Consider some merchandise the quality of which can be either good or bad. There are two types of consumers, connoisseur and gullible. Whilst it is certain that the connoisseur will judge the quality of the product correctly, the gullible layman will get it right only with probability $\frac{3}{4}$ (meaning he's wrong in $\frac{1}{4}$ of the cases). The connoisseur's opinion is more valuable than the layman's as

$$\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, in our example, β is given by

$$\beta = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

7.5 The Hirschleifer Paradox

We shall now examine the question whether information is always valuable. Suppose there are two states of the world occurring with probability p and $1-p$, respectively, and k consumers with initial endowments $\overline{x}_{k1}, \overline{x}_{k2}$. Now, define, U_k^* as

$$\max_{x_{k1}, x_{k2}} \{pu(x_{k1}) + (1-p)u(x_{k2})\} \equiv U_k^*$$

$$\text{s.t. } p_1 x_{k1} + p_2 x_{k2} \leq p_1 \overline{x}_{k1} + p_2 \overline{x}_{k2}$$

Individual rationality implies:

$$U_k^* \geq \overline{U}_k \equiv pu(\overline{x}_{k1}) + (1-p)u(\overline{x}_{k2})$$

i.e. nobody can be made worse off by voluntary trade.

Now, suppose there is a public signal before trading can occur, so that everybody knows the true state of the world. In that case, no trade will happen as nobody would be willing to trade income in the state that everybody knows will occur for income in a state that will not occur. Thus, the *ex ante*—value of the signal is negative.

Only private information is individually valuable, though it may not always be socially desirable, as other agents may draw inferences from the informed agent's behavior, which may eventually also lead to market breakdown.

However, if there are other ways than trade for society to react to the information, then all information may be socially valuable (e.g. flood warnings).

Some people may be averse to information for psychological reasons (e.g. genetic testing).

Commitment without prior information may also be a nice signaling device. Consider this hands—on example by Drèze: A man has two daughters, Ann and Barbara, one of whom will inherit a million \$. Peter is in love with Ann, but Ann is afraid lest Peter just be after the money. Eventually, the father will announce which of his daughters will inherit the million \$. Peter will prefer to propose before the father announces who will inherit, as marrying Ann with there being a chance of not getting the million \$ is a signal that is much less costly (if costly at all) for the type that loves Ann than for the type that loves money.

7.6 Search

- Suppose agent's (indirect) utility depends on some economic variable (p), such as price, wage, quality...
- This variable is randomly distributed, i.e. different stores are (unsystematically) charging different prices
- There is some (costly) mechanism for getting price information (magazines, browsing around in stores, making phone calls...)
- Search costs are dependent on technology (e.g. the internet leads to lower search costs), number of stores, opportunity costs for time (meaning wealthier individuals will search less).
- In our Walrasian models of Bertrand competition, we suppose there is perfect information, which is equivalent to saying that search costs are 0.

Example • Risk—neutral agent

- Agent's valuation:1
- Price uniformly distributed on the unit interval
- Marginal search costs of c
- 2 stores

(a) Search without recall

Backward induction: Suppose agent did not buy at the first store. Then, she should always buy at the second store. Her expected payoff in period 2 is thus: $E[\pi] = 1 - E[p] - c = \frac{1}{2} - c$. In period 1, agent receives a price offer of p_1. Thus, agent should buy in period 1 iff $1 - p_1 \geq \frac{1}{2} - c$. From this, we get agent's reservation prices $\hat{p}_1 = \frac{1}{2} + c$ and $\hat{p}_2 = 1$.

(b) With recall

In period 1, agent has learned p_1. Thus, the value of the additional information agent could glean in period 2 is given by

$$Pr[p_2 < p_1]E[p_1 - p_2 \mid p_2 < p_1] - c = p_1(p_1 - E[p_2 \mid p_2 < p_1]) = p_1 \cdot \frac{p_1}{2} - c \geq 0$$

$$\iff p_1^2 \geq 2c$$

Thus, we get agent's reservation prices $\hat{p}_1 = \sqrt{2c}$ and $\hat{p}_2 = p_1$, respectively.

Now, let us consider a more general case. Let prices be distributed according to the density function $f(p)$ and the cumulative distribution function $F(m) = Pr[p \leq m] = \int_0^m f(p)\, dp$. Let c continue to denote marginal search costs, whilst N is the total number of stores. Furthermore, let m_j denote the minimum price after j periods of searching. The expected gain from searching in one more store is then given by

$$Pr[p < m_j] \cdot E[m_j - p \mid p < m_j] - c$$

$$= E[\max\{0; m_j - p\}] - c$$
$$= \int_0^{m_j} (m_j - p) f(p)\, dp - c$$
$$= (m_j - p)F(p)|_0^{m_j} - \int_0^{m_j} (-1)F(p)\, dp - c$$
$$= \int_0^{m_j} F(p)\, dp - c \equiv g(m_j) - c$$

Hence, agent should stop searching when $g(m_j) \leq c$. The optimum stopping rule is thus a reservation price rule, with reservation price $\hat{p} = g^{-1}(c)$. Note that, apart from the last store, the reservation price is independent of the period. The reservation price in the last period is $\hat{p}_N = m_{N-1}$. Also note that as the decision problem is recursive, agent will only decide whether to go on searching for one more period, because, in our model, the information agent gleans in stores where she does not buy is of no value whatsoever.

- Optimum search behavior depends on the price distribution $f(p)$ and on the search costs.
- $c \uparrow \Rightarrow \hat{p}_t \uparrow$ since in optimum $c = \int_0^{\hat{p}} F(p)\, dp$
- Consider $G(p)$ with $F(p) \succ^{SOSD} G(p)$, meaning with $G(p)$, you are more likely to find very high (low) prices. SOSD implies that $\int_0^m G(p)\, dp \geq \int_0^m F(p)\, dp$. Thus, reservation price is lower with G, $\hat{p}_G < \hat{p}_F$, implying more searching going on with G, as you are more likely to find a really good deal if prices are very dispersed.
- Rothschild (1974) looks at a situation where consumers do not know the distribution of prices. His results are by and large similar to those presented here, but a lower price will point to a price distribution different from that agent had believed for thus far; reservation prices are thus not stable.

Thus far, we have supposed the distribution of prices to be exogenously given. But why would firms want to set different prices? Suppose all the firms and all the consumers were identical, then, an obvious equilibrium would be for all the firms to set $p = \hat{p}$. However, search costs enable firms to turn a profit. To induce search, however, there has to be price dispersion. Thus, in order to model search behavior, one needs

- heterogeneous firms
- *ex post*—heterogeneous consumers
- some mechanism that matches consumers and firms: *market matching models*

These market matching models, such as McKenna's in "Surveys in the Economics of Uncertainty" basically lead to the following conclusions:

- In equilibrium, there is no searching
- No firm will set a price above the consumers' reservation price
- Stemming from the specifics of MacKenna's model, many firms will set their prices equal to the consumers' reservation price (there is probability mass on $p = \hat{p}$).
- Key result: Firms do wield limited monopoly power, which may either be grounded regionally and appertain to the physical search costs (of going from store to store, e.g.) or it may be customer—specific (due to differing search costs, as consumers have differing opportunity costs for time e.g.)

Extensions of this model have been applied to the labor market, where employees search for better wage deals whilst still on the job. In this case, one observes a segmentation of the market: In equilibrium, firms are indifferent between paying higher wages and thus having a low turnover rate, and paying lower wages while suffering from a high turnover rate.

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8 Non Expected Utility Theory

In 1974, J.H. Drèze claimed: "In other words, a person who does not accept of the axioms of simple ordering for conditional acts, consequences and events, should not expect any assistance from scientific methods in handling decision problems." In this chapter, we shall examine if this is really so. First, we shall have a look at some paradoxes (8.1.) before briefly turning our attention to some newer developments in the literature that might help us handle one or the other of these paradoxes (8.2.).

8.1 Some Selected Problems of Expected Utility Theory

8.1.1 The Allais Paradox

Let there be two gambles, 1 and 2. In gamble 1, agent has to choose between a lottery A_1, yielding 1 million with probability 1 and a lottery B_1, paying out 5 million with a probability of 10%, 1 million with a probability of 89%, but 0 with a probability of 1%. In gamble 2, agent will choose between lottery A_2, yielding 1 million with a probability of 11%, and 0 with the counter—probability, and lottery B_2, paying out 5 million with a probability of 10%, 0 with a probability of 90%.

Most test persons chose A_1 and B_2 but

$$A_1 \succ B_1$$

$$\iff u(1) > 0,1u(5) + 0,89u(1) + 0,01u(0)$$

$$\iff 0,11u(1) > 0,1u(5) + 0,01u(0) \qquad (*)$$

whereas

$$B_2 \succ A_2$$

$$\iff 0,1u(5) + 0,9u(0) > 0,11u(1) + 0,89u(0)$$

$$\iff 0,1u(5) + 0,01u(0) > 0,11u(1)$$

which is in contradiction to (*). Thus, people seem systematically to behave in contradiction of the independence axion, which is seminal to expected utility theory (cf. Chapter 2). Indeed, define lottery $A \sim (1mil.; 1)$ and $B \sim (0, 5mil.; \frac{1}{11}, \frac{10}{11})$. Then,

$$A_1 \sim 0,11 \cdot A + 0,89 \cdot 1mil.$$

$$B_1 \sim 0,11 \cdot B + 0,89 \cdot 1mil.$$

$$A_2 \sim 0,11 \cdot A + 0,89 \cdot 0$$

$$B_2 \sim 0,11 \cdot B + 0,89 \cdot 0$$

Machina came up with an ingenious way to represent a three—dimensional lottery in a two—dimensional diagram by using the fact that $p_1 + p_2 + p_3 = 1$. Suppose the three possible payoffs are 0, 1 million, and 5 million.

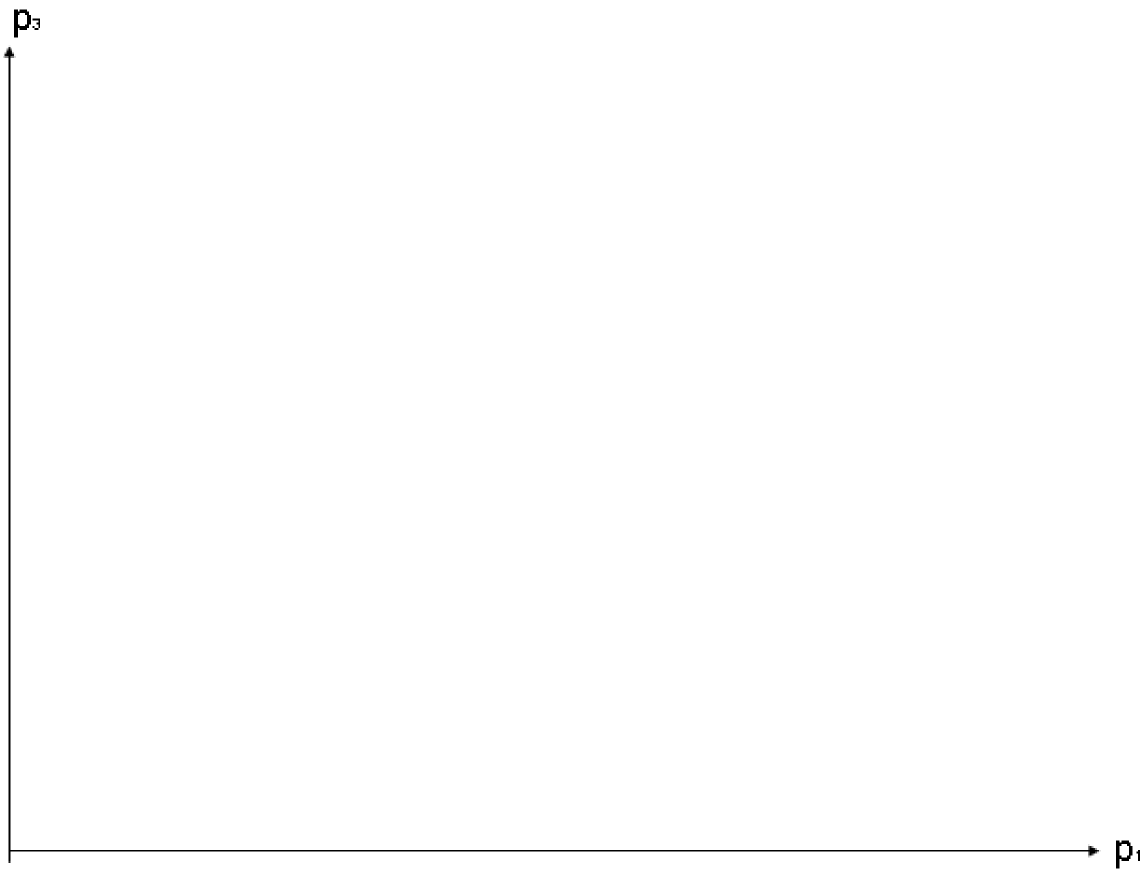

Figure 19: Machina's Triangle

If expected utility theory applies, indifference curves in Machina's Triangle have to be linear and parallel. Why?

$$Eu = p_1 u(0) + (1 - p_1 - p_3)u(1) + p_3 u(5)$$

$$dEu = dp_1[u(0) - u(1)] + dp_3[u(5) - u(1)] \stackrel{!}{=} 0$$
$$\iff \frac{dp_3}{dp_1} = -\frac{u(0) - u(1)}{u(5) - u(1)} = const$$

As a reaction, Machina proposed a "fanning out" of the linear indifference curves.

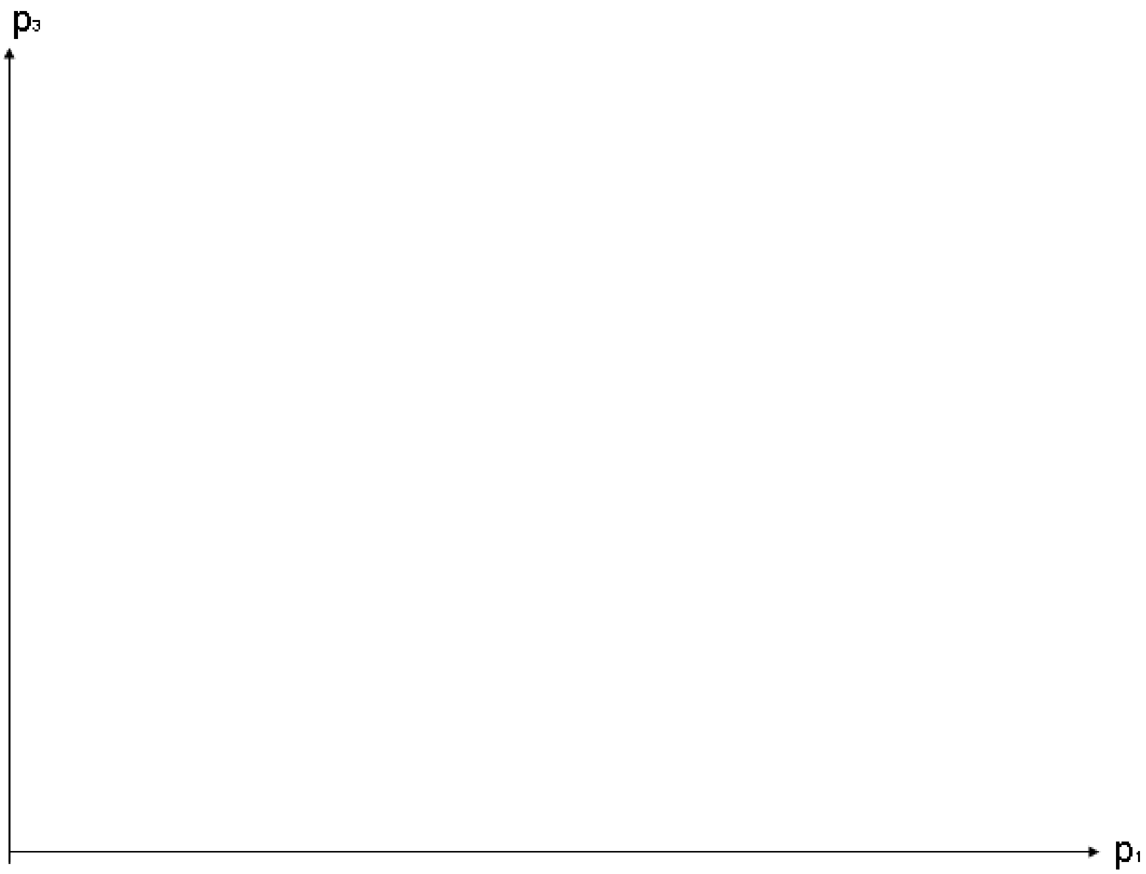

Figure 20: "Fanning Out" in Machina's Triangle

Other reactions:

- Hirschleifer & Riley: The question was misleading

- Substitue the independence axiom, e.g. by the "Betweenness Axiom": For all lotteries L and L':

$$\forall \lambda \in [0;1] : L \sim L' \Rightarrow \lambda L + (1-\lambda)L' \sim L$$

- New theory: "Regret Theory" (cf. class)

8.1.2 Ellsberg Paradox

Consider the following gamble: In box 1, there are 50 red balls and 50 black balls. In box 2, there are also 100 black and red balls, but the proportion of red to black balls is not known. Agent will choose between lottery A_1, which will pay out 100$ if the ball drawn from box 1 is red and 0 otherwise, and lottery B_1, which will pay out 100$ if the ball drawn from box 2 is red and 0 otherwise. Generally, people prefer A_1. Now, the test person is asked to choose between lottery A_2, which will pay out 100$ if the ball drawn from box 1 is black and 0 otherwise, and lottery B_2, which will pay out 100$ if the ball drawn from box 2 is black and 0 otherwise. Most of the people who have preferred A_1 over B_1 now prefer A_2 over B_2, which is inconsistent, because rational agents should only prefer A_2 over B_2 if they believe that there are fewer than 50 black balls in box 2. But that would imply that there are more than 50 red balls in box 2, so these people should then also prefer B_1 over A_1.

- People may be risk—averse over probabilities, meaning they prefer certain probabilities over uncertain probs. By contrast, vNM—utility is risk—neutral (linear) concerning probabilities.
- "Lemons" explanation: People may think the gamemaster has some additional information (e.g. that there are green balls in box 2)
- Hans—Werner Sinn has proved that for a certain class of utility functions it is indeed optimal to assign equal probability in case of uncertainty

8.1.3 Gambling

There are explanations of gambling that are consistent with expected utility theory:

- Differing subjective probabilities (e.g. horse racing)
- The entertainment aspect is more important than the financial aspect

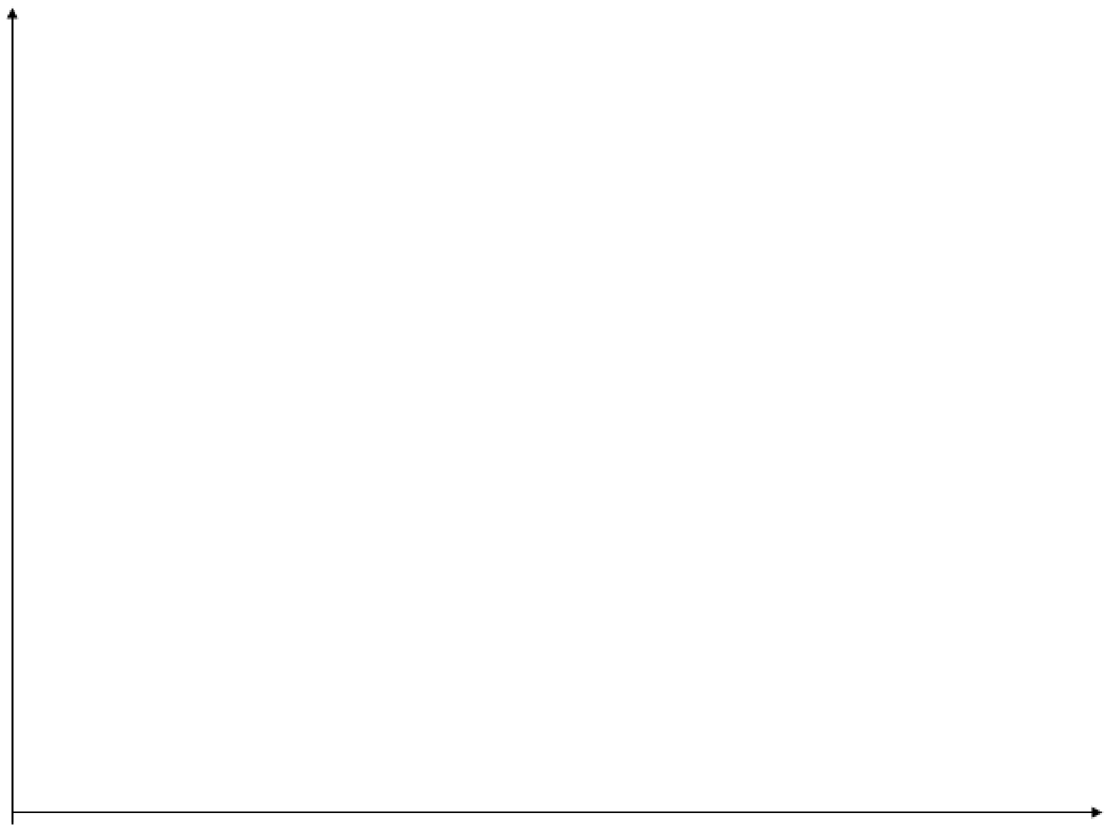

Figure 21: Friedman & Savage (1948)

Friedman & Savage thought that people's utility function was

- concave for low incomes: demand for insurance
- convex for medium incomes
- concave for high incomes: explains why lotteries would usually have several different prizes

Problem: There is too much gambling going on in this model, which seems more suited to pathological gamblers than to the general population

8.1.4 Some Further Problems of Expected Utility Theory

Most of these hark back to Kahnemann & Tversky:

- People do not seem to take previous information into account; there does not seem to be much in the way of Bayesian updating going on in the real world, which may spell trouble for the predictive value of Game—Theoretic analyses
- People do not have a feeling for the impact of sample size
- Many people think (h:heads; t:tails) hthtth likelier than ttthhh, and the latter likelier than tttttt
- Monty Hall's Puzzle
- Framing effects
- House money effects
- Endowment effects
- ...

8.2 Some Selected Reactions to the Problems With Expected Utility Theory

- Drèze: cf. beginning of the chapter
- Hirschleifer & Riley use an evolutionary argument: If people did not behave according to expected utility theory that would open up arbitrage possibilities for scoundrels and other people of the same sort to avail themselves of people's irrationality. Hence, Hirschleifer & Riley conclude that people behave as predicted by expected utility theory, at least when the stakes are high.

For the rest of this chapter, however, we shall be focusing on *non expected utility theory*, which generally consists in replacing vNM's independence axiom with something else, such as, e.g., Machina's fanning out. Other examples include:

- Weighted probabilities: Probabilities are assigned a weighting function ϕ s.t. $U(L) = \sum \phi(p_i)x_i$.

Now, suppose $\phi(0,5) < 0,5$ and $L = (0.5, 0.5; x + \epsilon_1, x + \epsilon_2)$. $U(L)$ is now given by $U(L) = \phi(0.5)(x + \epsilon_1) + \phi(0.5)(x + \epsilon_2) < x$ for $\epsilon_1, \epsilon_2 > 0$ and very small. However, lottery L dominates x as L will certainly lead to a payoff higher than x.

- Local Expected Utility Theory (Machina): People only locally behave as predicted by expected utility theory.
- Rank—Dependent Expected Utility (Quiggin)

We will now more closely examine "Prospect Theory" by Kahnemann & Tversky.

8.2.1 Prospect Theory (Kahnemann & Tversky, Econometrica, 1979)

Expected utility theory having been refuted, Kahnemann & Tversky were looking for a new *descriptive* concept. They were able to distinguish two phases in the decision process, an editing phase and a valuation phase:

1. *Editing Phase*

   - *Coding*: Link prospect, i.e. gamble, to a reference point
   - *Combination*: Pool identical outcomes
   - *Segregation*: A common riskless component is segregated from the prospect
   - *Cancelation*: Discard common constituents
   - *Simplification*: $(0.51, 0.49; 0, 99) \sim (0.5, 0.5; 0, 100)$
     Discard extremely unlikely outcomes
   - *Detection of Dominance*: Do not consider dominated prospects

2. *Valuation Phase*

   A prospect's value V is expressed on two scales, v and π:

   - Probabilities p are weighted with the function $\pi : p \mapsto \pi(p)$
   - $v : x \mapsto v(x)$ assigns a value to each outcome; it measures deviations from the reference point so that a change in the reference point may lead to a change in the preference order, *utility being defined on gains and losses, rather than wealth levels.*

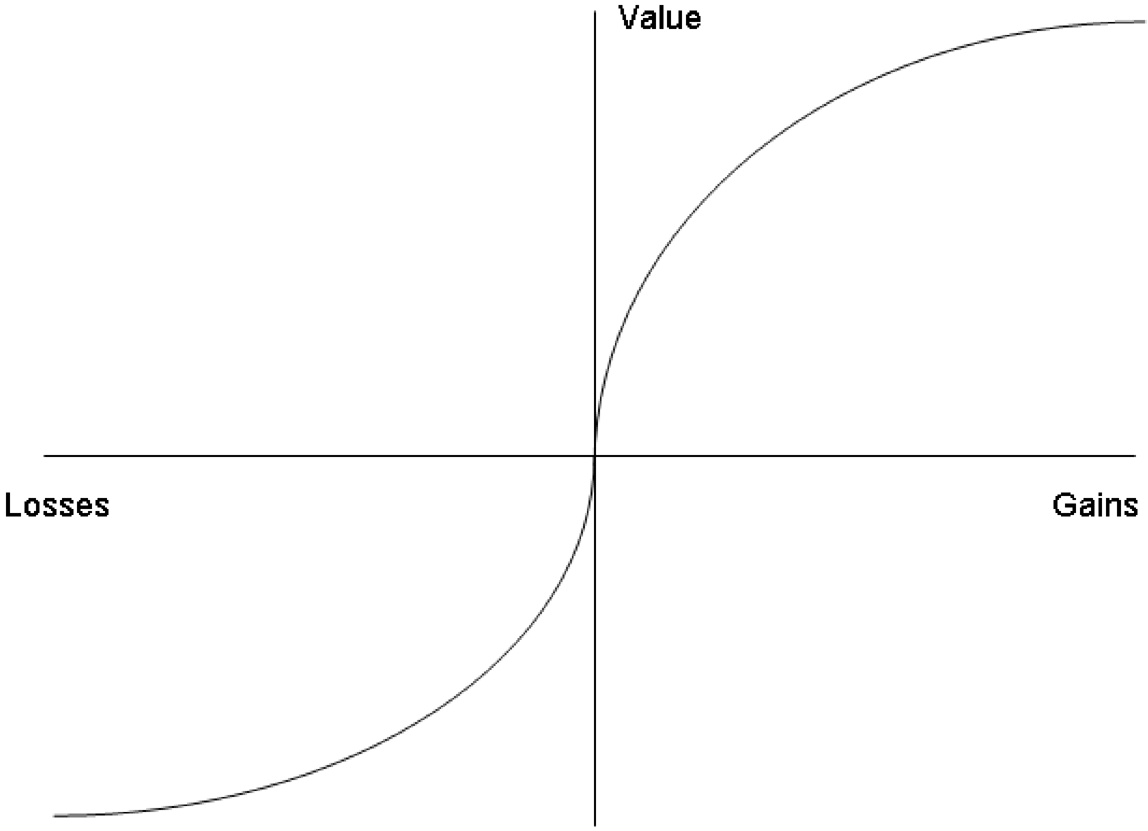

Figure 22: Shape of the value function

8.2.2 Hyperbolic Discounting

Here, the discount factor is no longer assumed to be constant, in contrast to exponential discounting, where for all $t \in [0; T]$ payments are discounted by a factor δ^t (δ const). Hyperbolic discounting can account for time inconsistencies, such as agent's preferring $2a$ in 101 days over a in 100 days, whilst preferring a today over $2a$ tomorrow.

8.2.3 Inequity Aversion

Consider the following ultimatum game:

Player 1, the proposer, has to decide which share $s \in [0; 1]$ of some sum of money X he would be willing to concede to player 2, the responder. The latter can either accept of or reject the former's offer. If he accepts, he gets a payoff of sX, whilst player 1 receives $(1-s)X$. If he rejects, neither player receives anything.

Backward induction would yield that 2 would be indifferent between accepting of a proposal of $s = 0$ and rejecting it. 1 will anticipate that and propose $s = 0$, of which 2 will accept in subgame—perfect Nash equilibrium (SPNE). If there is a smallest monetary unit, there is another SPNE where 2 gets that smallest monetary unit.

However, as e.g. Camerer & Thaler (1995) have shown, real—world players will usually not play the SPNE—strategies.

- The modal value of s is 0.5
- The average is $s \in [0.3; 0.4]$
- Offers $s < 0.2$ will usually be rejected
- The rejection rate is decreasing in s
- The rejection rate for a given s is lower if s is determined by random rather than deliberately chosen by player 1
- Cameron (1995) shows that these results do not change as the stakes are raised.