Chapter 1
The Demand for Insurance

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1 Introduction

The "demand for insurance" can first of all be interpreted as the demand for **cover.** Insurance is bought by means of a contract, which specifies a set of events, whose occurrence will create a financial loss for the buyer. The insurer undertakes to pay compensation (often also called an indemnity) in the event of these losses, and this is the cover. In exchange the buyer pays a premium for certain, usually at the time of entering into the contract. The essential characteristics of all insurance contracts are therefore: loss events, losses, cover and premium.

The details and complexity of specific insurance contracts will however vary greatly with the particular kinds of risks being dealt with. Though for theoretical purposes we model insurance as completely defined by the above four elements, we should recognise that in applications to specific markets, for example health, life, property and liability insurance, it will be necessary to adapt this general framework to the particular characteristics of the market concerned.

We can go beyond this descriptive account of the insurance contract, to obtain a deeper interpretation of the demand for insurance and of the economic role that insurance markets play. The effect of this interpretation is to place insurance squarely within the standard framework of microeconomics, and this has powerful analytical advantages, since it allows familiar and well-worked out methods and results to be applied. The basis of the approach is the concept of the **state of the world**. For present purposes, it is sufficient to define a state of the world as corresponding to an amount of loss suffered by the insurance buyer (later we will consider more general definitions). The situation in which she incurs no loss is one possible state, and there is then an additional state for each possible loss. The simplest case is that in which there is only one possible loss, so we have two states of the world. At the other extreme, losses may take any value in an interval $[0, L_{max}]$, in which case there is an uncountable infinity of possible states of the world, each defined by a point in the interval. We shall consider below models of both these cases, as well as others.

The key idea is that we can define the buyer's income in each state of the world, y,and call this her **state contingent income**, since the value it takes depends on which state of the world occurs. Before entering into an insurance contract, the consumer has given endowments of state contingent incomes, y_0 if no loss occurs, and $y_0 - L$ given the occurrence of loss L. If she buys insurance, she will receive under the contract an amount of compensation q that will generally depend on L, and will pay *for sure, i.e.* in every state of the world, a premium P. Thus with insurance her state contingent incomes become $y_0 - P$ in the no loss state, and $y_0 - L - P + q$ in the loss states. Then the important insight is that by allowing the buyer to vary P and q, the insurance market is effectively allowing her to vary her state contingent incomes away from those she is initially endowed with. *Insurance permits trade in state contingent incomes.* Moreover, these state contingent incomes can be interpreted as the "goods" in the standard model of the consumer, and the "demand for insurance" becomes, under this interpretation, the demand for state contingent incomes. Then, the insurance decision becomes a special case of a very well researched and understood model.

In the rest of this chapter, we shall find it useful to consider both concepts of the demand for insurance - the demand for cover, and the demand for state contingent incomes - side by side, since each gives its own insights and interpretations. Common to both is the basic microeconomic framework of optimal choice. The demand for insurance is viewed as the solution to the problem of maximising a utility function subject to a budget constraint. This utility function is taken from the leading theory of preferences under uncertainty, usually referred to as the *Expected Utility Theory.* The theory of insurance demand can be regarded as an application, indeed one of the most successful applications, of this theory. Under it, the consumer is modelled as having a von Neumann-Morgenstern utility function $u(y)$, which is unique up to a positive linear transformation and is at least three times continuously differentiable. Since y is income, we take the first derivative $u'(y) > 0$, more income is always preferred to less. Moreover, we assume that the insurance buyer is risk averse, and so $u^{''}(y) < 0$, the utility function is strictly concave. The sign of $u^{'''}(y)$, which defines the curvature of the marginal utility function $u^{'}(y)$, we leave open for the moment. Note that the utility function is the same regardless of whether we are in the no loss or loss state. That is, the utility function is *state independent*. This is not always an appropriate assumption for insurance, and we consider the effects of changing it below.

An important property of the utility function concerns whether its *Arrow-Pratt index of risk aversion*

$$A(y) \equiv -\frac{u^{''}(y)}{u^{'}(y)} \tag{1}$$

is increasing, constant or decreasing in y. We will usually consider all three cases.

Under this theory, given a choice set of alternative probability distributions of income, each of which induces a corresponding probability distribution of utilities, the decision taker chooses that distribution with the highest expected value of utility, hence the name. We now consider the insights into the demand for insurance this theory gives us.

2 Two Basic Models of the Demand for Insurance

The first step is to define the budget constraint appropriately, and then, formulating the problem as the maximisation of utility subject to this constraint, we can go on to generate the implications of the model. The simplest models have just two possible states of the world, a no loss state and a single state with loss L. The probability of this loss is π. Thus the expected value of income without insurance is

$$\bar{y} = (1-\pi)y_0 + \pi(y_0 - L) = y_0 - \pi L \tag{2}$$

with πL the expected value of income loss. We assume $L < y_0$.

Expected utility in the absence of insurance is

$$\bar{u}^0 = (1-\pi)u(y_0) + \pi u(y_0 - L) \tag{3}$$

Thus, in the absence of insurance the individual has an uncertain income endowment with an expected utility of $\bar{u}^0$.

The insurer offers cover q at a *premium rate* p, where p is a pure number between zero and one (as is a probability). The *premium amount* is $P = pq$. We assume that the buyer can choose any value of $q \geq 0$. The non negativity restriction says simply that the buyer cannot gamble on the occurrence of the loss event, and is a realistic restriction on insurance markets. The assumption that cover is fully variable may well not hold in a real insurance market (one may only be able to choose full cover, $q = L$, as in health insurance, or there may be an upper limit on cover $q_{max} < L$, as in auto insurance) but an important goal of the analysis is to understand why such restrictions exist, and so it is useful to begin by assuming the most general case of no restrictions (beyond non negativity) on cover. Other possibilities are considered below. Finally, it is convenient to express the premium as the product of cover and a premium *rate*. This is a common, but not universal, way of expressing insurance premia in reality, but of course a premium rate, the price of one monetary unit of cover, can always be inferred from values of P and q. The key point is the assumption that $p = P/q$ is constant and independent of q.

As discussed in the Introduction, we obtain alternative model formulations by defining demand in terms of cover, on the one hand, and state contingent income, on the other.

2.1 The q-Model

We assume the buyer chooses $q \geqslant 0$ to maximise expected utility

$$\bar{u} = (1-\pi)u(y_0 - P) + \pi u(y_0 - L - P + q) \tag{4}$$

subject to the constraint

$$P = pq \tag{5}$$

Clearly, the simplest way to solve this is to substitute from the constraint into the utility function and maximise

$$\bar{u}(q) = (1-\pi)u(y_0 - pq) + \pi u(y_0 - L + (1-p)q) \tag{6}$$

giving the first order (Kuhn-Tucker) condition

$$\bar{u}^{'}(q^*) = -p(1-\pi)u^{'}(y_0 - pq^*) + (1-p)\pi u^{'}(y_0 - L + (1-p)q^*) \leqslant 0 \quad q^* \geqslant 0 \quad \bar{u}_q q^* = 0 \tag{7}$$

Taking the second derivative of $\bar{u}(q)$ we have

$$\bar{u}^{''}(q) = p^2(1-\pi)u^{''}(y_0 - pq) + (1-p)^2\pi u^{''}(y_0 - L + (1-p)q) < 0 \tag{8}$$

where the sign follows because of the strict concavity of the utility function at all y. Thus expected utility is strictly concave in q, and the first order condition is both necessary and sufficient for optimal cover q^*.

The Kuhn-Tucker condition implies two cases:

Optimal cover is positive:

$$q^* > 0 \Rightarrow \frac{p}{1-p} = \frac{\pi u^{'}(y_0 - L + (1-p)q^*)}{(1-\pi)u^{'}(y_0 - pq^*)} \tag{9}$$

Optimal cover is zero:

$$q^* = 0 \Rightarrow \frac{p}{1-p} \geq \frac{\pi u^{'}(y_0 - L)}{(1-\pi)u^{'}(y_0)} \tag{10}$$

Taking the case $q^* > 0$ and rearranging (2.8) we have

$$u^{'}(y_0 - pq^*) = \frac{\pi}{p}\frac{(1-p)}{(1-\pi)}u^{'}(y_0 - L + (1-p)q^*) \tag{11}$$

from which it is easy to show that the following must hold:

$$\begin{aligned} p &= \pi \Leftrightarrow q^* = L & (12)\\ p &> \pi \Leftrightarrow q^* < L & (13)\\ p &< \pi \Leftrightarrow q^* > L & (14) \end{aligned}$$

We call the case in which $p = \pi$ the case of a **fair premium,** that where $p > \pi$ the case of a **positive loading**, and that where $p < \pi$ the case of a **negative loading**. We can then state these first results of the model as:

with a fair premium the buyer chooses full cover;
with a positive loading the buyer chooses partial cover;
with a negative loading the buyer chooses more than full cover.

Thus, using (2.10), we have

$$\begin{aligned} p &= \pi \Leftrightarrow u^{'}(y - pq^*) = u^{'}(y - L + (1-p)q^*) \Leftrightarrow y - pq^* = y - L + (1-p)q^* \Leftrightarrow q^* = L & (15)\\ p &> \pi \Leftrightarrow u^{'}(y - pq^*) < u^{'}(y - L + (1-p)q^*) \Leftrightarrow y - pq^* > y - L + (1-p)q^* \Leftrightarrow q^* < L & (16)\\ p &< \pi \Leftrightarrow u^{'}(y - pq^*) > u^{'}(y - L + (1-p)q^*) \Leftrightarrow y - pq^* < y - L + (1-p)q^* \Leftrightarrow q^* > L & (17) \end{aligned}$$

where the last two results follow from the fact that $u^{'}(.)$ is decreasing in its argument, *i.e.* from risk aversion.

Taking the case of zero cover, since risk aversion implies $u^{'}(y_0 - L) > u^{'}(y_0)$, p must be sufficiently greater than π for this case to be possible.

We illustrate these results in Figure 1. The five curves shown there graph the function $\bar{u}^{'}(q)$ for varying values of p, given the remaining parameters of the problem. The lower is p, the higher is the corresponding curve (we prove this below). The lowest two curves correspond to values of p sufficiently high that optimal q is zero. Otherwise q^* is given by the intersection of a curve with the q-axis.The negative slopes of the curves follow from (.). Note that in the case of the lowest curve, the buyer would really like negative cover, but this is not permitted. The three upper curves illustrate the cases of partial cover, full cover and more than full cover respectively.

Figure 1 about here

We can obtain an alternative diagrammatic representation of the equilibrium, which turns out to be useful in later applications of the analysis, as follows. Given the condition in (2.8), assume $q^* > 0$ and rewrite the condition as

$$\frac{\pi u^{'}(y_0 - L - P^* + q^*)}{(1-\pi)u^{'}(y_0 - P^*) + \pi u^{'}(y_0 - L - P^* + q^*)} = p \tag{18}$$

where $P^* = pq^*$ is the premium payment at the optimum. The reader should confirm that this is the condition that would be obtained by solving the problem of maximising expected utility in (.) with respect to P and q, and subject to the constraint in (.). We can interpret the ratio on the LHS of (.) as a marginal rate of substitution between P and q, i.e. as the slope of an indifference curve in (q, P)-space, and then this condition has the usual interpretation as the equality of marginal rate of substitution and price, or tangency of an indifference curve with a budget line.

This is illustrated in Figure 2. The lines show the constraint $P = pq$ for varying values of p. The indifference curves show (q, P)-pairs that yield given levels of expected utility. We shall justify the shape shown in a moment. Optimal q in each case is given by a point of tangency. For $p = \pi$, this point corresponds to L, as we have already established. The reader should illustrate a case in which $q^* = 0$.

Figure 2 about here.

It remains to justify the shapes of the indifference curves shown in Figure

2. Along any indifference curve in the (q, P)-space, we must have

$$\bar{u}(q, P) = (1 - \pi)u(y_0 - P) + \pi u(y_0 - L - P + q) = k \tag{19}$$

for some constant k. Using subscripts to denote partial derivatives, we have

$$\begin{aligned} \bar{u}_q &= \pi u^{'}(y_0 - L - P + q) && (20) \\ \bar{u}_P &= -[(1 - \pi)u^{'}(y_0 - P) + \pi u^{'}(y_0 - L - P + q)] && (21) \\ \bar{u}_{qq} &= \pi u^{''}(y_0 - L - P + q) && (22) \\ \bar{u}_{qP} &= \bar{u}_{Pq} = -\pi u^{''}(y_0 - L - P + q) && (23) \\ \bar{u}_{PP} &= \pi u^{''}(y_0 - L - P + q) + (1 - \pi)\pi u^{''}(y_0 - P) && (24) \end{aligned}$$

Then, from the Implicit Function Theorem, we have that the slope of an indifference curve is

$$\frac{dP}{dq} = -\frac{\bar{u}_q}{\bar{u}_P} > 0 \tag{25}$$

so this justifies the positive slopes of the indifference curves in Figure 2. Moreover,setting $q^* = L$ gives

$$\frac{dP}{dq} = \pi \tag{26}$$

so that at this point on the q-axis all indifference curves have the same slope, π.

To justify the curvature, consider first Figure 3. The characteristic of this curvature is that all points in the interior of the convex set formed by the indifference curve yield a higher level of expected utility than any point on the indifference curve. For example point A in the figure must yield a higher expected utility than point B because it offers the same cover for a lower premium. Since B and C yield the same expected utility, A must be better than C also. A function having this property is called *strictly quasiconcave.* Thus we have to prove that the function $\bar{u}(q, P)$ is strictly quasiconcave. The easiest way to do this is to show that $\bar{u}(q, P)$ is in fact strictly concave, because every strictly concave function is also strictly quasiconcave. $\bar{u}(q, P)$ is strictly concave if the following conditions are satisfied

$$\bar{u}_{qq} < 0 \tag{27}$$

$$\begin{vmatrix} \bar{u}_{qq} & \bar{u}_{qP} \\ \bar{u}_{Pq} & \bar{u}_{PP} \end{vmatrix} = \bar{u}_{qq}\bar{u}_{PP} - \bar{u}_{Pq}\bar{u}_{qP} > 0 \tag{28}$$

The first condition is satisfied, because of risk aversion. By inserting the above expressions for the second order partials and cancelling terms we obtain that the determinant is equal to

$$(1-\pi)\pi u^{''}(y_0 - L - P + q)u^{''}(y_0 - P) > 0 \tag{29}$$

as required. Intuitively, since the utility function is strictly concave in income, and income is linear in P and q, we can expect it to be strictly concave in these variables.

Figure 3 about here

To summarise the results so far: a risk averse insurance buyer who chooses cover to maximise expected utility, given a constant premium rate p, will choose full cover if this premium rate is equal to the loss probability, partial cover if premium rate exceeds loss probability, and more than full cover if premium rate is below loss probability. From the first order condition (.), we can solve for optimal cover as a function of the exogenous variables of the problem, income, the premium rate (price), the amount of loss, and the loss probability

$$q^* = q(y_0, p, L, \pi) \tag{30}$$

We call this function the buyer's *cover demand function*. We consider its main properties below.

2.2 The y-Model

We now let the choice variables in the problem be the state contingent incomes y_1 and y_2 respectively, where

$$y_1 = y - pq \tag{31}$$
$$y_2 = y - L + (1-p)q \tag{32}$$

The buyer's expected utility is now written as

$$\bar{u}(y_1, y_2) = (1-\pi)u(y_1) + \pi u(y_2) \tag{33}$$

An indifference curve corresponding to this expected utility function is shown in Figure 4. Since the function is strictly concave in incomes, it is strictly quasiconcave, and so the indifference curve has the curvature familiar from the standard model of the consumer. Its slope at a point, the marginal rate of substitiution, is given by

$$\frac{dy_2}{dy_1} = \frac{(1-\pi)u^{'}(y_1^*)}{\pi u^{'}(y_2^*)} \tag{34}$$

Note therefore that at a point on the certainty line OC, along which $y_1 = y_2$, this slope becomes equal to the probability ratio $-(1-\pi)/\pi$.

Figure 4 about here

Now, solving for q in (.), substituting into (.) and rearranging gives

$$(1-p)[y_0 - y_1] + p[(y_0 - L) - y_2] = 0 \tag{35}$$

or

$$(1-p)y_1 + py_2 = y_0 - pL \tag{36}$$

We can interpret this as a budget constraint, with $(1-p)$ the price of y_1, p the price of y_2, and $y_0 - pL$ "endowed wealth", a constant, given p. The point where $y_1 = y_0$, $y_2 = y_0 - L$ clearly satisfies this constraint. Thus we can draw the constraint as a line with slope $-(1-p)/p$, passing through the point $(y_0, y_0 - L)$, as shown in Figure 5. The interpretation is that by choosing $q > 0$, the buyer moves leftward from the initial endowment point $(y_0, y_0 - L)$, and, if there are no constraints on how much cover can be bought, all points on the line, including the *certain income,* y_C, are attainable. The price ratio or rate of exchange of the state contingent incomes is $-(1-p)/p$. The demand for insurance can now be interpreted as the demand for y_2, income in the loss state. Note that the budget line is flatter, the higher is p.

Figure 5 about here

The elimination of cover q to obtain this budget constraint in (y_1, y_2)-space is more than just a simple bit of algebra. It can be interpreted to mean that what an insurance market essentially does is to make available a budget constraint that allows the exchange of state contingent incomes: buying insurance means giving up income contingent on the no-loss state in exchange for income in the loss state, at a rate determined by the premium rate in the insurance contract.

Solving the problem of maximising expected utility in (.) subject to the budget constraint (.) yields first order conditions on the optimal state contingent incomes

$$\begin{aligned}(1-\pi)u^{'}(y_1^*) - \lambda(1-p) &= 0 & (37)\\ \pi u^{'}(y_2^*) - \lambda p &= 0 & (38)\\ (1-p)y_1^* + py_2^* &= y_0 - pL & (39)\end{aligned}$$

The first two can be expressed as

$$\frac{(1-\pi)u^{'}(y_1^*)}{\pi u^{'}(y_2^*)} = \frac{1-p}{p} \tag{40}$$

which has the interpretation of equality of the marginal rate of substitution with the price ratio, or tangency of an indifference curve with budget line. Writing this condition as

$$u^{'}(y_1^*) = \frac{\pi}{p}\frac{1-p}{1-\pi}u^{'}(y_2^*) \tag{41}$$

allows us to derive the results

$$\begin{aligned}p &= \pi \Leftrightarrow u^{'}(y_1^*) = u^{'}(y_2^*) \Leftrightarrow y_1^* = y_2^* & (42)\\ p &> \pi \Leftrightarrow u^{'}(y_1^*) < u^{'}(y_2^*) \Leftrightarrow y_1^* > y_2^* & (43)\\ p &< \pi \Leftrightarrow u^{'}(y_1^*) > u^{'}(y_2^*) \Leftrightarrow y_1^* < y_2^* & (44)\end{aligned}$$

Referring back to (.) and (.), equal state contingent incomes must imply full cover, a higher income in the no loss state must imply partial cover, and a higher income in the loss state must imply more than full cover. Thus we have the same results as before.

This solution is illustrated in Figure 6. Define the *expected value line* by

$$(1-\pi)y_1 + \pi y_2 = \bar{y} \tag{45}$$

This is clearly also a line passing through the initial endowment point. Recall that any indifference curve in (y_1, y_2)-space has a slope of $-(1-\pi)/\pi$ at the point at which it cuts the certainty line OC. Then clearly the cases of full, partial and more than full cover correspond to the cases in which the budget constraint defined by p is respectively coincident with, flatter than, or steeper than the expected value line (see the figure), since the coverage chosen, as long as it is positive, is always at a point of tangency between an indifference curve and a budget line. Note that if the budget line is so flat that it does not intersect the indifference curve passing through the initial endowment point, then we have the case where $q^* = 0$, the buyer stays at the initial endowment point.

Figure 6 about here

It is useful to be able to read off from the figure in state contingent income space the amount of cover bought. Figure 7 shows how to do this. Given the optimal point a, draw a line parallel to the certainty line. This therefore has a slope of 1, and cuts the line ce at b. Then the length be represents the cover bought. To see this note that $ed = pq^*$, while $bd = ad = (1-p)q^*$. So $be = bd + de = pq^* + (1-p)q^* = q^*$.

Figure 7 about here

The y-model allows us to solve for the desired state contingent incomes as functions of the exogenous variables of the problem

$$y_s^* = y_s(y_0, p, L, \pi) \quad s = 1, 2 \tag{46}$$

Thus we have demand functions for state contingent incomes as an alternative way, to that given by the cover demand function, of expressing the demand for insurance. The two models, the q-model and the y-model, are of course fully equivalent, and both are used frequently in the literature. The q-model is more direct and often easier to handle mathematically. The advantage of the y-model on the other hand is that it allows the obvious similarities with the standard consumer theory to be exploited, especially in the diagrammatic version. In the remainder of this book we will use whichever model seems more suitable for the purpose in hand.

3 Comparative Statics: The Properties of the Demand Functions

We want to explore the relationships between the optimal value of the endogenous variable, the demand for insurance, and the exogenous variables that determine it, y_0, p, π, and L. For an algebraic treatment, the q-model is more suitable, but we also exploit its relationship with the y-model to obtain additional insights.

Recall that the first order condition of the q-model, assuming $q^* > 0$, is

$$\bar{u}_q = -p(1-\pi)u'(y_0 - pq^*) + (1-p)\pi u'(y_0 - L + (1-p)q^*) = 0 \tag{47}$$

Applying the Implicit Function Theorem we have that

$$\frac{\partial q^*}{\partial y_0} = -\frac{\bar{u}_{qy_0}}{\bar{u}_{qq}} \tag{48}$$

$$\frac{\partial q^*}{\partial p} = -\frac{\bar{u}_{qp}}{\bar{u}_{qq}} \tag{49}$$

$$\frac{\partial q^*}{\partial L} = -\frac{\bar{u}_{qL}}{\bar{u}_{qq}} \tag{50}$$

$$\frac{\partial q^*}{\partial \pi} = -\frac{\bar{u}_{q\pi}}{\bar{u}_{qq}} \tag{51}$$

We have already shown that, because of risk aversion, $\bar{u}_{qq} < 0$. Thus the sign of these derivatives is in each case the same as that of the numerator.

To see the intuition for this, consider Figure 8. This shows an equilibrium with $q^* > 0$ as determined by the $\bar{u}_q$ curve. Now, if a change in an exogenous variable shifts the curve up, i.e. has a positive effect on $\bar{u}_q$ at the point q^*, then, given that the curve has a negative slope ($\bar{u}_{qq} < 0$), the value of q at which it cuts the horizontal axis must increase, and conversely if the change shifts the curve down. Thus we just have to identify the effect of a change in the value of an exogenous variable on the marginal expected utility of cover.

Figure 8 about here

The Effect of a Change in Income

We have that

$$\bar{u}_{qy_0} = -p(1-\pi)u^{''}(y_0 - pq^*) + (1-p)\pi u^{''}(y_0 - L + (1-p)q^*) \tag{52}$$

Consider first the case in which $p = \pi$ and so $q^* = L$. Inserting these values gives

$$\frac{\partial q^*}{\partial y_0} = -\frac{\bar{u}_{qy_0}}{\bar{u}_{qq}} = 0 \tag{53}$$

The reason is intuitively obvious. Since full cover is bought, and L stays unchanged, a change in income has no effect on insurance demand. More interesting is the case in which $p > \pi$ and so $q^* < L$. In that case, as the reader should check, we have $\bar{u}_{qy_0} \gtreqless 0$, *i.e.* the effect cannot be signed, insurance demand could increase or decrease with income.

This indeterminacy should not come as a surprise to anyone who knows standard consumer theory: income effects can typically go either way. Thus insurance cover can be an inferior or a normal good. It is however of interest to say a little more than this, by relating this term to the buyer's attitude to risk. To do this we make use of the y-model. Given the optimal incomes in the two states, we have $y_1^* > y_2^*$ because of partial cover. From the first order condition in the y-model we have

$$p(1-\pi) = \frac{(1-p)\pi u^{'}(y_2^*)}{u^{'}(y_1^*)} \tag{54}$$

Substituting this into (.) gives

$$\begin{aligned} \bar{u}_{qy_0} &= -u^{''}(y_1^*)\frac{(1-p)\pi u^{'}(y_2^*)}{u^{'}(y_1^*)} + (1-p)\pi u^{''}(y_2^*) &(55)\\ &= (1-p)\pi u^{'}(y_2^*)[\frac{u^{''}(y_2^*)}{u^{'}(y_2^*)} - \frac{u^{''}(y_1^*)}{u^{'}(y_1^*)}] &(56)\end{aligned}$$

Recall now the definition of the Pratt-Arrow measure of (absolute) risk aversion

$$A(y) \equiv -\frac{u^{''}(y)}{u^{'}(y)} \tag{57}$$

We can then write

$$\bar{u}_{qy_0} = (1-p)\pi u^{'}(y_2^*)[A(y_1^*) - A(y_2^*)] \tag{58}$$

Thus

$$\bar{u}_{qy_0} \gtreqless 0 \tag{59}$$

according as

$$A(y_1^*) \gtreqless A(y_2^*) \tag{60}$$

Since $y_1^* > y_2^*$, insurance cover is a normal good if risk aversion increases or is constant with income ($A(y_1^*) \geq A(y_2^*)$), and an inferior good if risk aversion decreases with income ($A(y_1^*) < A(y_2^*)$). Since the latter is what we commonly expect, the conclusion is that we expect that insurance is an inferior good. The intuition is straightforward: if an increase in income increases one's willingness to bear risk, then one's demand for insurance falls, other things equal.

This could be bad news for insurance companies: the demand for insurance could well be predicted to fall as incomes rise. It could also be bad news for the theory, since the evidence suggests that the opposite has happened. However, a resolution might well be tucked away in the "other things equal" clause. In reality, we would expect that as incomes rise, so does the value of the losses insured against. This is almost certainly true in health, life, property and liability insurance. As we now see, this increases the demand for insurance.

The Effect of a Change in Loss

We have that

$$\bar{u}_{qL} = -(1-p)\pi u^{''}(y_0 - L + (1-p)q^*) > 0 \tag{61}$$

Thus, as we would intuitively expect, given risk aversion, an increase in loss increases the demand for cover, *other things being equal.*

The Effect of a Change in Premium Rate

The effects of a price change on demand are always of central interest and importance. We have

$$\bar{u}_{qp} = -[(1-\pi)u^{'}(y_1^*) + \pi u^{'}(y_2^*)] + [p(1-\pi)u^{''}(y_1^*) - (1-p)\pi u^{''}(y_2^*)]q^* \tag{62}$$

This too cannot be unambiguously signed, since the first term is negative while the second could have either sign. But notice that the second term is just $-u_{qy_0}q^*$. In fact we have a standard Slutsky equation, which we can write as

$$\frac{\partial q^*}{\partial p} = -\frac{\bar{u}_{qp}}{\bar{u}_{qq}} = \frac{(1-\pi)u^{'}(y_1^*) + \pi u^{'}(y_2^*)}{\bar{u}_{qq}} + q^* \frac{u_{qy_0}}{\bar{u}_{qq}} \tag{63}$$

The first term is the substitution effect, and is certainly negative ($\bar{u}_{qq} < 0$), while the second is the income effect and, as we have seen, could be positive or negative (or zero). If $u_{qy_0} \geq 0$, this income effect is negative or zero, and so the demand for cover certainly falls as the premium rate (price) rises. That is, there is no ambiguity if absolute risk aversion increases or is constant with income. On the other hand, if insurance is an inferior good, the income effect is positive and so works against the substitution effect. That is, insurance may be a Giffen good if risk aversion decreases sufficiently with income.

The intuition is also easy to see, in terms of the y-model. A fall in the premium rate reduces the price of income in state 2 relative to that in state 1, and so, with utility held constant, y_2 will be substituted for y_1, implying an increased demand for cover. However, the fall in premium also increases real income, to an extent dependent on the amount of cover already bought, q^*, and this will tend to reduce the demand for insurance if risk aversion falls with income, and increase it if risk aversion increases with income.

The Effect of a Change in Loss Probability

$$\bar{u}_{q\pi} = pu^{'}(y_1^*) + (1-p)u^{'}(y_2^*) > 0 \tag{64}$$

Thus, as we would expect, an increase in the risk of loss increases demand for cover. Note, however, there is a strong "other things equal" assumption here. In general we would not expect the premium rate to remain constant when the loss probability changes, though we need some theory of the supply side of the market before we can predict how it would change.

4 Multiple Loss States and Deductibles

The simple two-state model considered so far in this chapter is useful, but of course limited. One aspect of this limitation is that the idea of "partial cover" is very simple: in the single loss-state, $q < L$. In reality, when there are multiple loss states, there can be different types of partial cover. One is the case of **coinsurance**, where a fixed proportion of the loss is paid in each state. The other is the case of a **deductible:** nothing is paid for losses below a specified value, called the deductible; when losses exceed this value, the insured receives an amount equal to the loss minus the deductible.

In practice, a deductible is a much more commonly observed form of partial cover than coinsurance. We now examine why this is. It can be shown that, when offered a choice between a contract with a deductible and any other contract with the same premium, assumed to depend only on the expected cost to the insurer of the cover offered, a risk averse buyer will always choose the deductible. This offers an explanation of the prevalence of deductibles and at the same time a confirmation of the predictive power of the theory.

We generalise the two-state model by assuming now that the possible loss lies in some given interval: $L \in [0, L_m]$, and has a given probability function $F(L)$ with density $f(L) = F'(L)$. Under coinsurance we have cover

$$q = \alpha L \quad \alpha \in [0,1] \tag{65}$$

with $\alpha = 0$ implying no insurance and $\alpha = 1$ implying full cover. Under a deductible we have

$$q = 0 \quad L \leqslant D \tag{66}$$
$$q = L - D \quad L > D \tag{67}$$

where D denotes the deductible, with $D = L_m$ implying no insurance and $D = 0$ full cover. The difference between the two contracts is illustrated in Figure 9, which shows cover as a function of loss.

Given the premium amount P, and an endowed income y_0 in the absence of loss, the buyer's state-contingent income in the case of coinsurance is

$$y_\alpha = y_0 - L - P + q = y_0 - (1-\alpha)L - P \tag{68}$$

and in the case of a deductible is

$$y_D = y_0 - L - P + q = y_0 - L - P + \max(0, L - D) \tag{69}$$

Figure 10 shows these incomes. The important thing to note about a deductible is that for $L \geqslant D$, the insurance buyer is fully insured at the margin: for losses above the deductible, her income becomes certain, and equal to

$$\hat{y}_D = y_0 - L - P + (L - D) = y_0 - P - D \tag{70}$$

It is this fact that accounts for the superiority, to a risk-averse buyer, of the deductible contract over other forms of contract with the same premium (and expected cost to the insurer). Under a deductible, income cannot fall below $\hat{y}_D$, however high the loss.

Figures 9 and 10 about here

Consider now the probability distribution function for the buyer's income under a given deductible contract. We have

$$\begin{aligned}
prob(y_D &\leqslant y^{'}) \;\; for \;\; y^{'} \in [\hat{y}_D, y_0 - P]) &(71)\\
&= prob(L \geqslant L^{'}) \;\; for \;\; L^{'} \in [0, D] &(72)\\
&= 1 - F(L^{'}) &(73)\\
prob(y_D &< \hat{y}_D) = 0 &(74)
\end{aligned}$$

This function $H(y)$ is illustrated in Figure 11. To the left of $\hat{y}_D$ it is just the horizontal axis, to the right it is determined by $F(L)$.

Now consider another type of contract with the same expected value of cover as the deductible contract in question, and therefore the same premium. This alternative could be a coinsurance contract, or any other kind of contract. We can show that it must have the kind of distribution function shown in the figure as $G(y)$, with the area *abc* equal to the area *cde*. But this then means that $H(y)$ is better than $G(y)$ in the sense of *second order* *stochastic dominance*. That is, $H(y)$ would be preferred to $G(y)$ by any risk averse buyer. $G(y)$ is riskier than $H(y)$, but has the same expected value.

To see that any alternative to the deductible contract must have the general properties of $G(y)$, note the following points:

- If the deductible contract and the alternative contract have the same expected cost, *i.e.* value of cover, to the insurer, they imply the same expected income to the buyer. The expected value of income under each of the contracts is $E[y_0 - L - P + q]$, and so, given that $y_0, E[L]$ and P are the same in each case, $E_G[q] = E_H[q]$ implies that the expected incomes are equal.

- It is a standard result that if two distributions with the same support have the same expected value, then the areas under the distribution functions are equal. Thus the areas under $H(y)$ and $G(y)$ in the figure must be the same.

- This implies that if a contract tries to improve on the deductible by having a lower distribution function to the right of D in the figure, it must have a higher distribution function to the left of D, and the corresponding areas must be the same.

  Figure 11 about here

The impressive aspect of this result is its generality and simplicity. However, it is also useful to consider the explicit solution for an optimal contract in the case of multiple loss states, allowing for the possibility that the premium may not be fair. The preceding discussion tells us what we should be looking for, in the case of a positive loading - a contract with a deductible.

It is simplest to use the q-model, and a finite number of loss states with losses, $L_1, ..., L_S$. We order the states so that $0 < L_1 < L_2 < ... < L_S$,with corresponding loss probabilities $\pi_s > 0$, and with $\pi_0 = 1 - \sum_{s=1}^{S} \pi_s > 0$ the probability of no loss. The premium amount P is proportional to the expected value of cover, and is given by

$$P = k \sum_{s=1}^{S} \pi_s q_s \tag{75}$$

with $k \geq 1$ (we ignore the case of a negative loading). The insurance buyer's expected utility is

$$\bar{u}(q_1, ..., q_S, P) = \pi_0 u(y_0 - P) + \sum_{s=1}^{S} \pi_s u(y_0 - L_s - P + q_s) \tag{76}$$

and she chooses cover in each state to maximise this, subject to the constraint defining the premium in (.) and the non negativity conditions

$$q_s \geq 0 \quad s = 1, ..., S \tag{77}$$

These constraints play an important role, as we will see.

Forming the Lagrange Function

$$L = \pi_0 u(y_0 - P) + \sum_{s=1}^{S} \pi_s u(y_0 - L_s - P + q_s) + \lambda(P - k \sum_{s=1}^{S} \pi_s q_s) \tag{78}$$

we have the Kuhn-Tucker conditions, which, (since $u^{''} < 0$), are both necessary and sufficient for an optimum

$$\frac{\partial L}{\partial q_s} = \pi_s u^{'}(y_0 - L_s - P^* + q_s^*) - \lambda^* k \pi_s \leq 0 \quad q_s^* \geq 0 \quad q_s^* \frac{\partial L}{\partial q_s} = 0 \tag{79}$$

$$\frac{\partial L}{\partial P} = -[\pi_0 u^{'}(y_0 - P^*) + \sum_{s=1}^{S} \pi_s u^{'}(y_0 - L_s - P^* + q_s^*)] + \lambda^* = 0 \tag{80}$$

$$\frac{\partial L}{\partial \lambda} = P^* - k \sum_{s=1}^{S} \pi_s q_s^* = 0 \tag{81}$$

where an asterisk denotes an optimal value. We assume that cover in at least one state is positive, so that the premium is positive, otherwise the problem is uninteresting. This implicitly puts an upper bound on k, and means that $\frac{\partial L}{\partial q_s} = 0$ for at least one s.

Intuitively, we expect that with $k = 1$, we will have full cover, *i.e.* $q_s^* = L_s$, for all s , while for $k > 1$, we will obtain a contract with a deductible. We now derive these results formally.

A first and most useful result is the following. Suppose, as we assume, we have $q_s^* > 0$ for some s. Then for every higher loss state $s + t$, with $t = 1, .., S - s$, we must have $q_{s+t}^* > 0$. This is true whatever the value of k.

The proof is by contradiction. Suppose $q_s^* > 0$ but $q_{s+t}^* = 0$ for some $t = 1, .., S - s$. Then from the conditions we have

$$u^{'}(y_0 - L_s - P^* + q_s^*) = \lambda^* k \geq u^{'}(y_0 - L_{s+t} - P^*) \tag{82}$$

By strict concavity of utility this implies

$$y_0 - L_s - P^* + q_s^* \leq y_0 - L_{s+t} - P^* \tag{83}$$

implying

$$L_{s+t} \leq L_s - q_s^* \tag{84}$$

which contradicts the fact that $L_s < L_{s+t}$.

Take now the case in which $k = 1$, so the premium is fair. If we could assume that all $q_s^* > 0$, then the full cover result is easy to derive. Thus, the condition on each q_s^* is

$$u^{'}(y_0 - L_s - P^* + q_s^*) = \lambda^* \tag{85}$$

implying that marginal utility in each loss state is equal. Inserting this into condition (.) then gives

$$-[\pi_0 u^{'}(y_0 - P^*) + \lambda^* \sum_{s=1}^{S} \pi_s] + \lambda^* = 0 \tag{86}$$

implying

$$\pi_0 u^{'}(y_0 - P^*) = (1 - \sum_{s=1}^{S} \pi_s)\lambda^* = \pi_0 \lambda^* \tag{87}$$

and therefore

$$u^{'}(y_0 - P^*) = u^{'}(y_0 - L_s - P^* + q_s^*) \tag{88}$$

which holds if and only if $q_s^* = L_s$, for all s.

However, we should prove, rather than assume, that cover is positive in each loss state in this case. Again we use a proof by contradiction. Suppose for the first t loss states cover is zero, and for the remaining states positive. Then the conditions become

$$u^{'}(y_0 - L_s - P^*) \leq \lambda^* \quad s = 1, .., t \tag{89}$$

$$u^{'}(y_0 - L_s - P^* + q_s^*) = \lambda^* \quad s = t+1, .., S \tag{90}$$

Note that these conditions imply that for any loss state $s^{'} \leq t$, we must have

$$L_{t+1} - q_{t+1}^{*} \geq L_{s'} > 0 \tag{91}$$

We can write condition (.) as

$$\pi_0 u^{'}(y_0 - P^{*}) + \sum_{s=1}^{t} \pi_s u^{'}(y_0 - L_s - P^{*})] = (1 - \sum_{s=t+1}^{S} \pi_s)\lambda^{*} = (\pi_0 + \sum_{s=1}^{t} \pi_s)\lambda^{*} \tag{92}$$

implying

$$\pi_0[u^{'}(y_0 - P^{*}) - \lambda^{*}] = \sum_{s=1}^{t} \pi_s[\lambda^{*} - u^{'}(y_0 - L_s - P^{*})] \tag{93}$$

We know from conditions (.) and (.) that the RHS of this equation must be non negative, and so we have

$$u^{'}(y_0 - P^{*}) \geq \lambda^{*} \tag{94}$$

implying

$$y_0 - P^{*} \leq y_0 - L_{t+1} - P^{*} + q_{t+1}^{*} \tag{95}$$

or $L_{t+1} - q_{t+1}^{*} \leq 0$. But this contradicts (.). This implies all $q_s^{*} > 0$, implying in turn, as we just saw, full cover in every loss state.

The intuition for this full cover result is essentially the same as that in the case of a single loss state. When faced with a fair premium, the buyer wants to equalise marginal utilities of income across all states, including the no-loss state. Since marginal utility is not state dependent, this implies equal incomes across all states, which in turn implies full cover in every loss state.

Turning now to the case in which $k > 1$, the proof we just gave, to the effect that cover must be positive in all states, no longer goes through. We now have the first order conditions

$$u^{'}(y_0 - L_s - P^{*}) \;\leq\; k\lambda^{*} \quad s = 1, .., t \tag{96}$$

$$u^{'}(y_0 - L_s - P^{*} + q_s^{*}) \;=\; k\lambda^{*} \quad s = t+1, .., S \tag{97}$$

where t is as before the highest loss state in which cover is zero. This again must imply $L_{t+1} - q_{t+1}^{*} \geq L_{s'} > 0$ for any loss state $s^{'} \leq t$. However, condition

(.) now yields

$$\pi_0 u^{'}(y_0 - P^*) + \sum_{s=1}^{t} \pi_s u^{'}(y_0 - L_s - P^*)] = (1 - k \sum_{s=t+1}^{S} \pi_s)\lambda^* < (\pi_0 + \sum_{s=1}^{t} \pi_s)\lambda^* \tag{98}$$

from which it is no longer possible to obtain a contradiction. Indeed we should not expect one: we know that with a positive loading there will be partial cover, and that this takes the form of a deductible, and we now have to show how this follows from the conditions.

First note that in the loss states $s = t+1, .., S$ in which cover is positive, we have

$$u^{'}(y_0 - L_{t+1} - P^* + q^*_{t+1}) = ... = u^{'}(y_0 - L_S - P^* + q^*_S) = k\lambda^* \tag{99}$$

and therefore

$$y_0 - L_{t+1} - P^* + q^*_{t+1} = ... = y_0 - L_S - P^* + q^*_S \tag{100}$$

implying

$$L_{t+1} - q^*_{t+1} = ... = L_S - q^*_S = D^* \tag{101}$$

where D^* is the optimal deductible. The intuition here is that where cover is positive, it is chosen to equalise marginal utilities across these states, implying equal incomes across these states, and this in turn implies a constant difference between loss and cover, *i.e.* a deductible.

Secondly, as we already saw, in the first t states, the fact that cover is zero implies

$$L_1 < ... < L_t \leq L_{t+1} - q^*_{t+1} = D^* \tag{102}$$

i.e. zero cover implies that the loss in these states is less than the deductible.

Note that in this discrete model, these two aspects of a deductible, that of a constant difference between loss and cover in positive cover states, and loss less than deductible in no-cover, lower loss states, are separable, in the sense that though the former will always hold, the latter may not. Thus it is possible, for L_1 sufficiently large, that $q_s > 0$, all s. In that case, condition (.) implies

$$\pi_0 u^{'}(y_0 - P^*) = [1 - k(1 - \pi_0)]\lambda^* \tag{103}$$

This shows that $k > 1$ rules out the equalisation of marginal utilities between the no-loss and loss states, and implies the partial cover result, while the equalisation of marginal utilities across loss states implies that this takes the form of a deductible. Thus it is straightforward to show that

$$k > 1 \Leftrightarrow \frac{1 - k(1 - \pi_0)}{\pi_0} < 1 \tag{104}$$

and so

$$u^{'}(y_0 - P^*) < k\lambda^* = u^{'}(y_0 - P^* - D^*) \tag{105}$$

implying

$$D^* > 0 \tag{106}$$

In a model in which losses lie in an interval $[0, L_m]$, the two aspects of a deductible inevitably occur together, since for any positive deductible, there is always an interval of loss values sufficiently close to zero that lies below it. When the lowest possible loss is however strictly positive, it could always happen that cover would be positive in all states. Remember that insurance is all about transferring income from states with relatively low expected marginal utilities of income to states with relatively high expected marginal utilities of income. If the state with the lowest loss is nevertheless one with a relatively high marginal expected utility of income, cover could be positive in all states even if there is a deductible.

Finally, note that if the constraint $q_s \geq 0$ is non-trivially binding at the optimum, this implies that the insurance buyer would be better off if she could actually choose negative cover in that state, *i.e.* make a payment to the insurer if that state occurs. The intuition is that that would allow a reduction in premium, and an increase in cover in higher loss states, and so a further transfer of income from low loss, low marginal utility of income states, to high loss, high marginal utility of income states. An interesting question, but one we do not explore here, is why this is not allowed under typical insurance contracts[1].

5 Insurance Demand with State Dependent Utility

It seems reasonable to believe that for at least some types of losses for which insurance can be bought, the utility of income will depend on whether or

[1]But see the later discussion of the Raviv model in Chapter X below.

not a particular event takes place, where this event may or may not also cause an income loss. Sickness is an obvious example. The utility of a given amount of income if one is sick may well differ from that if one is healthy, while sickness may cause a loss of employment income, medical costs and so on. A formal extension of the model of insurance demand to this case is quite straightforward, particularly if we revert to the case of a single loss state.

We again take state 1 as the no-loss state and state 2 as the loss-state, but now denote the utility function in state $s = 1, 2$ as $u_s(y)$, with $u_1(y^{'}) > u_2(y^{'})$, for all $y^{'} > 0$. Otherwise these are standard von Neumann-Morgenstern utility functions. For simplicity we shall always assume that insurance is offered at a fair premium, since this brings out clearly the implications of state dependent utilities. Extension to other cases is left as an exercise. Let p therefore denote both the probability of loss and the premium rate. Formulating the insurance problem as one of choosing state contingent incomes (the y-model), we have to solve

$$max_{y_1 y_2}(1-p)u_1(y_1) + pu_2(y_2) \quad s.t. \;\; (1-p)y_1 + py_2 = \bar{y} \tag{107}$$

where $\bar{y}$ is the expected value of income, determined by the endowed incomes in the no-loss and loss states respectively. Assuming an interior solution, it is easy to see that the optimum requires

$$u_1^{'}(y_1^*) = u_2^{'}(y_2^*) \tag{108}$$

At a fair premium, the insurance buyer will always want to equalise marginal utilities of income across states. However, this implies equality of *incomes* across states if and only if the marginal utility of income is not state dependent, which is something of a special case. More generally, we want to see what this condition of equality of marginal utilities implies for the choice of incomes, and therefore of insurance cover, across states, when the utility of income is state dependent.

We can distinguish three senses in which we could talk of "full insurance":

- choice of cover that equalises marginal utilities of income across states
- choice of cover that equalises total utilities of income across states
- choice of cover that equalises income across states.

When utility is state independent and the premium is fair, these three coincide: choice of cover equalises incomes, marginal and total utilities. Under state dependent utility, as we have just seen, marginal utilities will be equalised, but it remains an open question whether incomes and total utilities are equalised. To explore this further, we have to find an economically meaningful way of relating the state dependent utility functions to each other.

A nice way of doing this was developed by P J Cook and D A Graham. At every income level y, assume there is an amount of income $w(y)$ that satisifies

$$u_1(y - w(y)) = u_2(y) \tag{109}$$

We could define $w(y)$ as the consumer's maximal willingness to pay to be in the "good" state 1 rather than the "bad" state 2. The notation emphasises that this willingness to pay may depend on the income level. Figure 12 illustrates this in the utility-income space. In the figure, for any given level of y in state 2, $w(y)$ gives the reduction in this income level required to yield an equal level of utility in state 1. It is just the horizontal difference between the two curves. This is a useful way to describe the relationship between the curves as y changes.

Figure 12 about here

To develop this further, since (.) is an identity, differentiating through with respect to y gives

$$u_1'(y - w(y))[1 - w'(y)] = u_2'(y) \tag{110}$$

or

$$w'(y) = 1 - \frac{u_2'(y)}{u_1'(y - w(y))} \tag{111}$$

Thus the way in which the willingness to pay changes as income varies is determined by the slopes of the utility functions at equal utility values. It seems reasonable to assume $w'(y) \geqslant 0$. For example, we would expect the willingness to pay to be healthy rather than sick at least not to fall with income. Thus we have

$$\begin{aligned} w'(y) &= 0 \Rightarrow u_2'(y) = u_1'(y - w(y)) \Rightarrow u_2'(y) > u_1'(y) \qquad (112) \\ w'(y) &> 0 \Rightarrow u_2'(y) < u_1'(y - w(y)) \Rightarrow u_2'(y) \gtreqless u_1'(y) \qquad (113) \end{aligned}$$

for all y. The figure illustrates the case of $w'(y) = 0$.

To see the effects of state dependent utility on the insurance decision, given the restriction in (.), we move to the state contingent income space in Figures 13 and 14. In the case where utility is not state dependent, we regard the 45^0 line as the certainty line, because equality of incomes implies equality of utilities. In the state dependent utility case, the 45^0 line still corresponds to certainty of *income*, but it no longer implies certainty of *utility*. A point on this line implies that utility in state 2 is below that in state 1 (refer back to Figure 12). In order to determine a locus of points at which utility across states is equal, *i.e.* certain, we know from (.) that we have to subtract $w(y)$ from each income level in state 2, the bad state, to obtain the income level in state 1, the good state, that yields the same utility level. Where $w^{'}(y) = 0$, this implies the line shown as WW in Figure 13, whereas when $w^{'}(y) > 0$ we have the curve $WW^{'}$ in Figure 14.

To analyse the insurance decision, take first the case shown in Figure 13. The initial incomes are as shown at point A, and the line passing through this point has slope $-(1-p)/p$. The optimality condition, given the fair premium, is that state contingent incomes after insurance cover is chosen must satisfy $u_1^{'}(y_1^*) = u_2^{'}(y_2^*)$. But if $w^{'}(y) = 0$, (.) shows that we must have

$$u_2^{'}(y_2^*) = u_1^{'}(y_2^* - w(y_2^*)) \tag{114}$$

implying

$$y_1^* = y_2^* - w(y_2^*) \tag{115}$$

The tangency between budget constraint and indifference curve must take place on the line WW in Figure 13, because marginal utilities of income are equal along this line. Then there is full insurance of utilities, in the sense that $u_1(y_1^*) = u_2(y_2^*)$, i.e. utilities are equalised across states. But there is more than full insurance of incomes, since $y_2^* > y_1^*$.

If $w^{'}(y_1^*) > 0$, the optimum cannot lie on $WW^{'}$ in Figure 15, because along that curve the marginal utility of income in state 2 is less than that in state 11. An optimal point must lie on the budget line to the right of where it intersects $WW^{'}$. Thus utility remains less than fully insured, in the sense that $u_1(y_1^*) > u_2(y_2^*)$. That is all that can be said without making further assumptions about the relation between marginal utilities of income in the two states. Three cases are possible, as the figure illustrates:

(a) $u_2^{'}(y) = u_1^{'}(y)$ at each y, so *marginal* utilities are state independent. Then the optimum is at α, where income is fully insured;

(b) $u_2'(y) > u_1'(y)$, so that at a given income, increasing income increases utility more in the bad state than in the good. Then the corresponding indifference curve through α must be flatter than the budget line, and the optimum must be at a point such as β, where more than full income insurance is bought;

(c) $u_2'(y) < u_1'(y)$, so that at a given income, increasing income increases utility more in the good state than in the bad. Then the corresponding indifference curve through α must be steeper than the budget line, and the optimum is at a point like γ, where less than full income insurance is bought.

An interesting implication of this analysis is that an insurance contract that restricts cover to the loss actually incurred - actual loss on income from employment, actual medical costs, in the case of health insurance - is optimal only if marginal utility of income is state independent.

Figures 13, 14 and 15 about here

6 Insurance Demand and Incomplete Markets

Up until now, it has been assumed that the insurance buyer faces only one type of loss against which insurance can be bought. In reality insurance markets are typically **incomplete**, in the sense that not all risks an individual faces can be insured against. Thus one can buy insurance against income loss arising from ill health, but not against income loss due to fluctautions in business conditions leading to loss of overtime, short-time working, and loss of bonuses. In other words, part of one's income may be subject to "background risk" which cannot be insured against. We now want to examine, in the simplest possible model, the effect the existence of an uninsurable risk can have on the purchase of insurance against an insurable risk, as well as the question of whether a welfare loss arises from the absence of a market for insurance against one of the risks. We know that the absence of a market cannot make the insurance buyer better off - one can always choose not to use a market if it is not optimal to do so. The question is whether the consumer is thereby made strictly worse off.

Suppose an individual has an income y_0, and faces a loss L with probability π and a loss K with probability θ. There are then four possible states of the world, with associated incomes set out in the following table. It is assumed that insurance cover q can be bought against risk of loss L at premium rate $p \geqslant \pi$. We are interested in the effect of the non-insurability of loss K on the buyer's choice of q.

Loss	0	L
0	$y_1 = y_0 - pq$	$y_3 = y_0 - L + (1-p)q$
K	$y_2 = y_0 - pq - K$	$y_4 = y_0 - L + (1-p)q - K$

The important point to note is that since only L can be insured against, it is possible to use the insurance market to transfer income only between **sets of states,** but not between all individual states**.** Insurance allows income to be exchanged between states 1 and 2, on the one hand, and 3 and 4 on the other, but not between 1 and 2, or between 3 and 4.

Denote the probability of state $s = 1, .., 4$ by ϕ_s. Clearly, since these four states are mutually exclusive and exhaustive, $\sum_s \phi_s = 1$. The exact values of these probabilities ϕ_s will depend on the nature of the stochastic relationship between the two losses. We consider here the three extreme cases:

(i) the two losses are statistically independent. In that case:
$\phi_1 = (1-\pi)(1-\theta)$ - neither loss occurs
$\phi_2 = (1-\pi)\theta$ - only K occurs
$\phi_3 = \pi(1-\theta)$ - only L occurs
$\phi_4 = \pi\theta$ - both losses occur

(ii) the two losses are perfectly positively correlated - either both occur or both do not occur. In effect then, there is only one loss, $L + K$, which for some reason can only be partially insured against. Then
$\phi_1 = (1-\pi) = (1-\theta)$ - neither loss occurs
$\phi_2 = \phi_3 = 0$ - we cannot have only one of the losses occurring
$\phi_4 = \pi = \theta$ - both losses occur

(iii) the losses are perfectly negatively correlated - if one occurs the other does not, and conversely. Then
$\pi = (1-\theta)$, $\theta = (1-\pi)$,
$\phi_1 = \phi_4 = 0$
$\phi_2 = \theta$
$\phi_3 = \pi$

The buyer will choose cover to solve

$$\max_q \bar{u}(q) = \sum_{s=1}^{4} \phi_s u(y_s) \quad s.t. \;\; q \geqslant 0 \tag{116}$$

given the specific expressions for y_s in the Table. The general form of the first order condition will be the same for cases (i) - (iii), but the interpretation will of course depend on the precise interpretation of the probabilities ϕ_s, which varies across the three cases. The first order (Kuhn-Tucker condition) is

$$\begin{aligned} \bar{u}_q &= -p[\phi_1 u'(y_1^*) + \phi_2 u'(y_2^*)] + (1-p)[\phi_3 u'(y_3^*) + \phi_4 u'(y_4^*)] \leqslant 0 \quad (117) \\ q^* &\geqslant 0 \quad \bar{u}_q q^* = 0 \quad (118) \end{aligned}$$

It is straightforward to show that the second order condition is satisfied.

The condition shows that if $q^* > 0$,

$$\frac{\phi_1 u'(y_1^*) + \phi_2 u'(y_2^*)}{\phi_3 u'(y_3^*) + \phi_4 u'(y_4^*)} = \frac{(1-p)}{p} \quad (119)$$

Thus the marginal rate of substitution on the left hand side has to be defined with reference to marginal utilities of income averaged over each subset of states within which state contingent incomes can **not** be exchanged. This is simply because an increase in q reduces incomes in **both** states 1 and 2 and increases incomes in **both** states 3 and 4. In order to exchange incomes between states within a subset we would require an insurance market for the loss K. We now want to see what effect the presence of the non-insurable risk has on the purchase of cover against the insurable risk.

Case (i), independence.

Writing in the explicit expressions for the incomes y_s^* and probabilities ϕ_s we obtain from the first order condition

$$\begin{aligned} \frac{\pi(1-p)}{p(1-\pi)} &\leqslant \frac{(1-\theta)u'(y_0 - pq^*) + \theta u'(y_0 - pq^* - K)}{(1-\theta)u'(y_0 - L + (1-p)q^*) + \theta u'(y_0 - L + (1-p)q^* - K)} \quad (120) \\ q^* &\geqslant 0 \quad \bar{u}_q q^* = 0 \quad (121) \end{aligned}$$

We now have to distinguish between two subcases:

(a) Fair premium, $p = \pi$. Then it is easy to see that $q^* = L$, we have full cover. Thus the background risk makes no difference to the optimal cover against L. To see this, note that the left hand side of the condition becomes 1 in this case. If $q^* < L$, the denominator in the right hand ratio must (because $u'' < 0$) be greater than the numerator, thus the ratio must be < 1 and the condition cannot be satisfied. If however $q^* = L > 0$ the ratio on the right hand side is 1 and equals the left hand side. If $q^* > L$, the numerator on the right hand side is larger than the denominator and the condition is not satisfied. Intuitively, one might think that, when insurance against L is available at a fair premium, one might over-insure, to compensate for not being able to insure against K. In the independence case this intuition is false, because it simply results in expected marginal utility across the states in which L does occur becoming smaller than that across the states in which L does not occur.

(b) Positive loading, $p > \pi$. In that case the ratio on the left hand side becomes $\rho < 1$. Then in that case $q^* = L$ cannot be optimal, because we just saw that the right hand ratio would then equal 1. Assume that $L > q^* > 0$, *i.e.* the loading is not so large that no cover is bought. We want to know what effect on choice of cover introduction of the risk K makes. In general, the answer depends on the precise form of the buyer's utility function. In fact we can show the following:

Introducing K, suitably small, increases cover, if and only if absolute risk aversion decreases with income;

Introducing K, suitably small, reduces cover, if and only if absolute risk aversion increases with income;

Introducing K, suitably small, leaves cover unchanged, if and only if absolute risk aversion is constant.

Proof: We prove only the first, the others follow similarly. Note first that if we want to *increase* the ratio on the right hand side of (.), we have to *increase* q^*, since this *reduces* both incomes and *increases* both marginal utilities in the numerator, and *increases* both incomes and *reduces* both marginal utilities in the denominator.

Now consider the equilibrium in the absence of the risk K. This has to satisfy the condition

$$\rho = \frac{u'(y_0 - pq^*)}{u'(y_0 - L + (1-p)q^*)} \tag{122}$$

We know then, that when we introduce K, since this leaves ρ unchanged, if this reduces the value of the ratio on the right hand side, we will have to increase q^* to restore equality. It is easy to show that the value of the ratio will be reduced (and cover therefore increased) if

$$\frac{u'(y_0 - pq^*)}{u'(y_0 - L + (1-p)q^*)} > \frac{u'(y_0 - pq^* - K)}{u'(y_0 - L + (1-p)q^* - K)} \tag{123}$$

that is, if

$$\frac{u^{'}(y_0 - L + (1-p)q^* - K)}{u^{'}(y_0 - L + (1-p)q^*)} > \frac{u^{'}(y_0 - pq^* - K)}{u^{'}(y_0 - pq^*)} \tag{124}$$

For short, write this as

$$\frac{u^{'}(y_3^* - K)}{u^{'}(y_3^*)} > \frac{u^{'}(y_1^* - K)}{u^{'}(y_1^*)} \tag{125}$$

Now assume that K is sufficiently small that it is permissible to use the simple Taylor series approximations

$$u^{'}(y_s^* - K) \approx u^{'}(y_s^*) - u^{''}(y_s^*)K \quad s = 1, 3 \tag{126}$$

Inserting these into (.) and cancelling terms then gives

$$A(y_3^*) \equiv -\frac{u^{''}(y_3^*)}{u^{'}(y_3^*)} > -\frac{u^{''}(y_1^*)}{u^{'}(y_1^*)} \equiv A(y_1^*) \tag{127}$$

Since $y_3^* < y_1^*$(partial cover), this gives the result.

Case (ii): *perfect positive correlation.*

In this case we can show that ideally, if there is fair insurance the buyer would like to set $q^* = L + K$, i.e. over-insure on the L-market to compensate for not being able to insure against K. If $p > \pi$, the buyer would like to set $q^* < L + K$, for reasons with which we are already familiar, and so we just consider the case of a fair premium. Introducing the appropriate probabilities and incomes for this case into the first order condition gives

$$\frac{(1-\pi)u^{'}(y_0 - pq^*)}{\pi u^{'}(y_0 - L + (1-p)q^* - K)} = \frac{1-p}{p} \tag{128}$$

implying

$$\frac{u^{'}(y_0 - pq^*)}{u^{'}(y_0 - L + (1-p)q^* - K)} = 1 \tag{129}$$

(Note, we can rule out the case in which $q^* = 0$ because then the ratio on the left hand side is strictly less than one, which does not satisfy the Kuhn-Tucker condition). Clearly then this condition is satisfied if and only if $q^* = L + K$. This is then a case in which the noninsurability of K does not reduce welfare, though it does change behaviour. However if, for some reason, cover is restricted in the L-market, for example by $q \leqslant L$, then the buyer chooses $q^* = L$ and is made strictly worse off by the non existence of the K-market.

Case (iii): *perfect negative correlation.*

Inserting the appropriate probabilities and incomes into the first order condition gives

$$\frac{(1-\pi)u'(y_0 - pq^* - K)}{\pi u'(y_0 - L + (1-p)q^*)} \geqslant \frac{1-p}{p} \tag{130}$$

We take the fair premium case, in which the condition becomes

$$u'(y_0 - pq^* - K) \geqslant u'(y_0 - L + (1-p)q^*) \tag{131}$$

Suppose first that $q^* > 0$, so the condition must hold with equality. This then implies

$$pq^* + K = L - (1-p)q^* \tag{132}$$

or

$$q^* = L - K \tag{133}$$

Now L and K are exogenous with $L \gtreqless K$. Thus we have three possibilities:

(a) $L = K$. This implies $q^* = 0$, which is a contradiction. In fact in this case no cover is bought. The reason is that, because of the perfect negative correlation and the equality of K and L, income is certain with zero insurance cover.

(b) $L > K$. Then $q^* = L - K > 0$. In order to equalise incomes across the states, cover has to be bought which just makes up the difference between L and K.

Note a feature of these two cases: the introduction of the second risk K certainly makes a difference to the insurance decision on the purchase of cover on the market for insurance against L, but, because of the perfect negative correlation, there is no welfare loss arising from the absence of a market for insurance against K.

(c) $K > L$. Then we would have $q^* < 0$, which is assumed not to be possible, and again contradicts the assumption that $q^* > 0$. In fact in this case we have $q^* = 0$: buying positive cover would worsen the income inequality between the two states, since it reduces income in the state in which K occurs and L does not. The buyer would actually like to have negative cover, i.e. offer a bet on the occurrence of the loss L, since this would transfer income from the state in which L occurs to that in which K occurs. In this case also, the insurance decision on the L-market is certainly affected by the existence of the non insurable risk K. The buyer would be made better off if the K-market existed and the L-market did not.